Decomposition of moduli spaces of algebras using deformation theory

Alice Fialowski

University of Pécs and Eotvos Loránd University Budapest

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Moduli spaces of algebras

The *Moduli Space* of $\mathbb{K}$-algebras of a fixed dimension n is the set of algebra structures on $V = \mathbb{K}^n$ modulo the action of the group $\mathbb{G}L(V)$.

Since the set of algebra structures on V is cut out by a set of quadratic equations, it is already an algebraic variety, so the structure of the moduli space of algebras is complicated.

With Michael Penkava we have studied moduli spaces of low dimensional complex Lie, super, differential graded and L_∞ algebras, including $\mathbb{Z}_2$-graded algebras.

In every one of these cases, the moduli space decomposes into very simple strata, consisting of projective orbifolds of the form $\mathbb{P}^n/G$, where G is a subgroup of the symmetric group Σ_{n+1}, which acts on $\mathbb{P}^n$ by permuting the projective coordinates.

Orbifold structure, stratification, deformations

- *Orbifold structure*: locally looks like the quotient space under the linear action of a finite group.
- *Stratification*: decomposition of a space into disjoint subsets which gives a partition of the space. It is useful when each strata are defined by some recognisable set of conditions and fit together manageably.
- *Jump deformation*: if there exists a 1-parameter family of deformations of the Lie algebra structure s.t. every nonzero value of the parameter determines the same deformed Lie algebra, which is not the original one.
- *Smooth deformations*: they move along the family, meaning that the Lie algebra structure is different for each value of the parameter.

The structure of the moduli space

- Complex projective spaces with the possible action of the symmetric group, and some exceptional points. The orbifold points, which are fixed by some element in the acting group, are special.
- The moduli space is glued together by deformations, which determine the elements that one may deform to locally. The exceptional points play a role in refining the picture of how this space is glued together.
- The different strata are connected by jump deformations.

L_∞ algebras

- strongly homotopy or sh Lie algebras - natural generalizations of Lie algebras, Lie superalgebras, and differential graded Lie algebras from a homotopy point of view.
- V: $\mathbb{Z}$- or $\mathbb{Z}_2$-graded vectorspace;
  L_∞ algebra on V: collection of n-ary multilinear maps
  - graded skew-symmetric
  - satisfy the generalized Jacobi identities, are fully characterized by their Chevally-Eilenberg differential graded algebras. The Jacobi identity is allowed to hold up to higher coherent homotopy.
- If $V = \oplus V_i, i \in \mathbb{Z}, V_i = 0$ for every $i \neq 0$, Jacobi identity, so they are the Lie algebras.

Conjecture

This type of decomposition happens for all such moduli spaces of finite dimensional Lie, super, L_∞ and associative algebras over $\mathbb{C}$.

- In this talk I will demonstrate the picture in 3 and 4 dimensional Lie algebras, and give an explicit construction of a stratification of part of the moduli space of Lie algebras of a given dimension in exactly this form, which holds in any finite dimensional space.

A simple example, 3-d Lie algebras

- Classical decomposition of moduli space of complex 3-d Lie algebras known since at least Jacobson (1962).
- A new stratification of the space.
  Later a complete projective picture of the moduli space, using cohomology and deformation computations.
- Final picture is the following. There are three strata, $d_2 = \mathfrak{r}_3(\mathbb{C})$, $d_2(p:q)$, parametrized by $\mathbb{P}^1/\Sigma_2$, and $d_3 = \mathfrak{sl}(2,\mathbb{C})$.
- We got this picture by computing cohomology and deformations.

Comparison of Classical Decomposition and Ours

- Complex Lie algebras except for $\mathfrak{sl}(2,\mathbb{C})$ are determined by equivalence classes of 2×2-matrices under similarity and up to multiplication by a nonzero constant. Thus Jordan decomposition is important.
- The classical picture decomposes these matrices as follows:

name	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$\mathfrak{r}_3(\lambda)$	$\mathfrak{r}_3(\mathbb{C})$	$\mathfrak{n}_3$
	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- Our picture decomposes these matrices as follows:

name	$d_2(p:q)$	d_3
	$\begin{bmatrix} p & 1 \\ 0 & q \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- For most cases, we have $\mathfrak{r}_3(\lambda) = d_2(p:q)$ where $\lambda = p/q$. The major difference is that we exchange the elements $\mathfrak{r}_3(\mathbb{C})$ and $\mathfrak{r}_3(1)$, that is, we switch which element belongs to the family!

It turns out that the switch above is *necessary* in order for the decomposition of the strata to be consistent with cohomology and deformations. Moreover, $\mathfrak{n}_3 = d_2(0:0)$.

Geometric picture

- $d_2(1:1)$ is $\mathfrak{r}_3(\mathbb{C})$ in the classical notation.
- The nilpotent algebra $\mathfrak{n}_3$ sits inside the solvable family with parameters $(0:0)$.
- The orbifold points of the projective family are $(1:1)$ and $(1:-1)$. At $d_2(1:1)$ there is a doppenganger $d_2 = \mathfrak{r}_3(1)$ whose neighborhoods coincide with those of the points $d_2(1:1)$ and which also deforms infinitesimally into $d_2(1:1)$. At $d_2(1:-1)$ there is a deformation in the d_3 direction as well, as a deformation in the direction of the family.
- d_3 is $\mathfrak{r}_3(1)$, which is the simple Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.
- $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$ is just $d_2(1:0)$.
- the members of the projective family $d_2(p,q)$ deform smoothly along the family.
- $d_2(0:0)$ has deformations everywhere except to d_2.

Higher Dimensional Generalizations

For 3-dimensional Lie algebras, we have $d_2(p:q)$ depends on the coordinates $(p:q)$ projectively. Moreover $d_2(p:q) \sim d_2(q:p)$, meaning that the stratum is parameterized by $\mathbb{P}^1/\Sigma_2$. This stratification corresponds to the classical observation that $\mathfrak{r}_3(\lambda) \sim \mathfrak{r}_3(1/\lambda)$.

- The pattern observed for 3-dimensional algebras replicates itself in higher dimensions.
- For the moduli space of $n+1$-dimensional complex Lie algebras we have a portion of the moduli space which is classified by similarity classes of $n \times n$ matrices up to multiplication of the diagonal elements by a nonzero constant.
- For the moduli space of 4-dimensional complex Lie algebras, the strata given by extending a 1-d trivial algebra by a 3-d trivial one are:

name	$d_5(p:q:r)$	$d_6(p:q)$	d_7
	$\begin{bmatrix} p & 1 & 0 \\ 0 & q & 1 \\ 0 & 0 & r \end{bmatrix}$	$\begin{bmatrix} p & 0 & 0 \\ 0 & p & 1 \\ 0 & 0 & q \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The Moduli Space of 4-d Lie Algebras I

- The singleton d_7 could be considered as the nontrivial element in the space $d_7(p)$ given by the matrix $\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$, which is parametrized by $\mathbb{P}^0$. Of course, this space has only 2 elements, and the element $d_7(0)$ is just the trivial algebra.
- $d_6(p:q)$ is parametrized by $\mathbb{P}^1$, without any action of the symmetric group.
- $d_5(p:q:r)$ is parametrized by $\mathbb{P}^2/\Sigma_3$.
- The deformations of these algebras within this group of algebras can be read from the matrices.
- Each matrix corresponds to a partition of the number 3. Thus $d_5(p:q:r)$ corresponds to $[1,1,1]$, $d_5(p:q)$ corresponds to $[2,1,0]$ and $d_7(p)$ corresponds to $[3,0,0]$. The number of distinct strata is the number of partitions of 3.

Moduli space of 4-d Lie Algebras II

- So the part of the moduli space given by extending a 1-d trivial algebra by a 3-d trivial one are described by those three types of matrices, up to scaling the diagonal elements.
- What about other strata? There are algebras which arise as extensions of the trivial 1-dimensional algebra by the only nontrivial nilpotent 3-d algebra $\mathfrak{n}_3$. The matrices representing the module structure on these algebras give another projective family of Lie algebras and a singleton algebra.
- Adding 2 more singletons coming from direct sums, altogether we have 3 projective families and 4 singletons.

5-dimensional Lie algebras

- Similarity types in dimension 4 give 5 dimensional Lie algebras:

$$d_{20} \begin{bmatrix} p & 1 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 0 & r & 1 \\ 0 & 0 & 0 & s \end{bmatrix} \quad d_{21} \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 1 & 0 \\ 0 & 0 & q & 1 \\ 0 & 0 & 0 & r \end{bmatrix} \quad d_{22} \begin{bmatrix} p & 1 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & p & 1 \\ 0 & 0 & 0 & q \end{bmatrix} \quad d_{23} \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 1 \\ 0 & 0 & 0 & q \end{bmatrix} \quad d_{24} \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$

- The first stratum $d_{20}(p : q : r : s)$ is parametrized by $\mathbb{P}^3/\Sigma_4$.
- The second stratum $d_{21}(p : q : r)$ is parametrized by $\mathbb{P}^2/\Sigma_2$, where Σ_2 acts by permuting the coordinates q and r.
- The third stratum $d_{22}(p : q)$ is parametrized by $\mathbb{P}^1/\Sigma_2$.
- The fourth stratum $d_{23}(p : q)$ is parametrized by $\mathbb{P}^1$.
- The fifth stratum $d_{24}(p)$ is parametrized by $\mathbb{P}^0$.

General Picture of this Stratum

It is easy to write down a matrix corresponding to a partition of n which represents a certain $n+1$-dimensional Lie algebra. Consider a multiindex of the form $[m_1, ..., m_n]$, $m_1 + \cdots + m_n = n$, $m_i \geq m_{i+1} \geq 0.$. Then each such multiindex determines a stratum. First, consider the case $m_1 > m_2$. Then the matrix should have $m_1 - m_2$ columns with only a p_1 on the main diagonal. For the $(m_1 - m_2 + 1)$-th row, there will be a 1 to the right of the entry p_1. Next, suppose that $m_2 = m_3 = ...m_k > m_{k+1}$. Then there will be $m_k - m_{k+1}$ repetitions of the pattern where the columns have a 1 above the entry on the main diagonal except in the first column, followed by the entries on the main diagonal given by $p_1, ..., p_k$ sequentially, repeating the pattern $m_k - m_{k+1}$ times. If we think that this has reduced all the entires in the multiindex to m_{k+1}, then we repeat the process again, until finally we have run out of nonzero entries in the multiindex.

Example

For example, consider the partition $[3, 2, 2, 1, 0, 0, 0]$ of 8. The matrix representing a certain stratum of 8-dimensional Lie algebras is

$$\begin{bmatrix} p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_4 \end{bmatrix}$$

The stratum in the moduli space of 8-dimensional complex Lie algebras corresponding to this matrix is parametrized by $\mathbb{P}^3/\Sigma_2$, where the action of Σ_2 is by permuting the coordinates p_2 and p_3.

The Generic Element

- The understanding that strata of moduli space of algebras are naturally parametrized by projective orbifolds goes back to the early 2000s, but the idea that we needed to consider the *generic element* in $\mathbb{P}^n$ arose later.
- One can define $\mathbb{P}^n = \mathbb{C}^{n+1}/\mathbb{C}^*$, but usually, people consider only $\mathbb{C}^{n+1}\backslash 0/\mathbb{C}^*$, which gives a complex manifold.
- The point $(0 : \cdots : 0)$, is known to algebraic geometers, who give it the misleading name, the generic element, even though it is far from generic. In fact, every nongeneric element in $\mathbb{P}^n$ lies in every open neighborhood of the generic element, so including it creates a terrible topology.
- The point $d(0 : \cdots : 0)$ in a projective family $d(p_1 : \cdots : p_n)$ of algebras behaves just like its counterpoint in algebraic geometry, meaning that this generic point has *jump deformations* to every other element in the family, reflecting somehow the idea that they are infinitesimally close to the generic point.

More on Generic Elements

- In the 3-d complex Lie algebras $d_2(0:0)$ is the nilpotent algebra $\mathfrak{n}_3$, so the nilpotent algebra sits at the top of a family of solvable (but not nilpotent) algebras.
- In the 4-d complex Lie algebras, $d_5(0:0:0)$ and $d_6(0:0)$ represent the two nontrivial nilpotent algebras, where $d_7(0)$ is the trivial Lie algebra.
- In the 4-d complex Lie algebras, the generic elements in the solvable families give nilpotent algebras, but there is a nilpotent Lie algebra which is not an element of a family.
- There is also a 4-d projective family $d_3(p:q)$ of solvable Lie algebras which are not of the types above, arising from extensions of the trivial lie algebra by $\mathfrak{n}_3$. Its generic element, $d_3(0:0)$, is isomorphic to $d_5(0:0:0)$.
- In general, the generic elements in two different strata can coincide, or even that the generic element in one family is an ordinary element in another family. These are the only possible overlaps.

Constructing Lie Algebras by Extensions

- Any Lie algebra V which is not semisimple, arises as an extension of an algebra W by another algebra M. In other words, there is an exact sequence

$$0 \to M \to V \to W \to 0.$$

- If V is not solvable, then there is a Levi decomposition, where M is solvable and W is semisimple, and in characteristic 0, we even can express $V = M \rtimes W$, a semidirect product.
- If V is solvable but not nilpotent, we can decompose it in the form where M is the maximal nilpotent ideal, and W is a trivial algebra (over $\mathbb{C}$ at least).
- If μ is the algebra structure on M and δ is the algebra structure on W, then the algebra structure d on V is of the form $d = \delta + \mu + \lambda + \psi$, where $\lambda : M \otimes W \to M$ is called the *module structure* and $\psi : \bigwedge^2(W) \to M$ is called the *cocycle*.

The Conditions for an Extension

There are three conditions necessary for a structure $d = \delta + \mu + \lambda + \psi$ to give a Lie algebra structure:

1. $[\mu, \lambda] = 0$. (The compatibility condition)
2. $[\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0$. (The Maurer-Cartan condition)
3. $[\delta + \lambda, \psi] = 0$. (The cocycle condition).

- The classical Maurer-Cartan condition is $[\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] = 0$.
- This holds when $[\mu, \psi] = 0$, in particular, if $\mu = 0$, which is the classic construction of an extension by a module.
- When $[\mu, \psi] \neq 0$, then λ is not really a module structure.
- When λ is a module structure, then the *cocycle condition* really implies that ψ is a cocycle with respect to the Lie algebra structure $\delta + \lambda$ on $M \oplus W$.

Reference

F., Penkava: Stratification of moduli spaces of Lie algebras, similar matrices and bilinear forms, J. Algebra 2018

THANK YOU FOR YOUR ATTENTION!