On the classification of Kantor triple systems[1]

Antonio Ricciardo
University of Bologna

Non-associative algebras in Cádiz
23rd February 2018

[1]Based on joint work with *N. Cantarini* and *A. Santi*.
https://arxiv.org/abs/1710.05375

Kantor triple systems

Definition (Jordan triple system)

A Jordan triple system (JTS) is a vector space with a ternary product commutative in the 1st and 3rd variables, i.e. $(xyz) = (zyx)$, and satisfying

$$(uv(xyz)) = ((uvx)yz) - (x(vuy)z) + (xy(uvz)) \quad (*)$$

The Kantor tensor is $K_{xz}(y) = (xyz) - (zyx)$. Commutativity in the 1st and 3rd variables is then equivalent to K being identically 0.

Definition

A Kantor triple system (KTS) is a vector space with ternary product satisfying condition $(*)$ and such that

$$K_{K_{uv}(x)y} = K_{(yxu)v} - K_{(yxv)u} \tag{1}$$

Example of KTS

Example (Special linear KTS)

The matrix space $M = M_{m,n}(\mathbb{C}) \oplus M_{r,m}(\mathbb{C})$ with product

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) = \begin{pmatrix} x_1 y_1^t z_1 + z_1 y_1^t x_1 - y_2^t x_2 z_1 \\ x_2 y_2^t z_2 + z_2 y_2^t x_2 - z_2 x_1 y_1^t \end{pmatrix} \tag{2}$$

is a KTS. Furthermore, it is

- simple, i.e. it has no non-trivial ideals (subsystems I for which $(IMM) + (MIM) + (MMI) \subset I$),
- centerless, i.e. $C = \{c \in M | (xcy) = 0, \ \forall x, y \in M\} = 0$.

Construction of KTS

Definition (Graded Lie algebras)

A Lie algebra $\mathfrak{g}$ is $\mathbb{Z}$-graded if $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
We say that $\mathfrak{g}$ is 5-graded (resp. 3-graded) if $\mathfrak{g}_i = 0$ for $|i| > 2$ (resp. $|i| > 1$).
If σ is an order 2 automorphism of $\mathfrak{g}$ we say that σ is a grade-reversing involution if $\sigma(\mathfrak{g}_i) = \mathfrak{g}_{-i}$.

Proposition

Let $(\mathfrak{g}, \sigma)$ be a pair consisting of a 5-graded Lie algebra $\mathfrak{g}$ and a grade-reversing involution σ. We define a KTS structure over $\mathfrak{g}_{-1}$ by

$$(xyz) = [[x, \sigma(y)], z] \tag{3}$$

Problem: Is it possible to reverse this construction?

Lie algebras and KTS

$$\left\{ \begin{array}{c} \text{simple} \\ \text{5-graded Lie alg.} \\ \text{with involution} \end{array} \right\} \quad \begin{array}{c} \longrightarrow \\ \longleftarrow \\ TKK \end{array} \quad \left\{ \begin{array}{c} \text{K-simple} \\ \text{KTS} \end{array} \right\}$$

TKK construction

Let V be a centerless KTS and define the operators, for $x, y, z \in V$:

$$L_{x,y}(z) := (xyz)\,, \quad \varphi_x(y) := L_{y,x}\,, \quad D_{x,y}(z) := -\varphi_{K_{x,y}(z)}\,.$$

$$\begin{array}{ccccccccccc} \mathfrak{g} = \mathfrak{g}(V) & = & \langle K_{x,y} \rangle & \oplus & V & \oplus & \langle L_{x,y} \rangle & \oplus & \langle \varphi_x \rangle & \oplus & \langle D_{x,y} \rangle \\ & & -2 & & -1 & & 0 & & 1 & & 2 \end{array}$$

The Lie bracket is defined by

$$[x, y] := K_{x,y}\,, \qquad [A, x] := A(x)$$

$$[x, y] := K_{x,y}\,, \qquad [A, x] := A(x)$$

for A either $L_{x,y}$, ϕ_x or $D_{x,y}$ and extended using

- transitivity (if $A \in \mathfrak{g}_i$, $i \geq 0$ and $[A, x] = 0\ \forall x \in \mathfrak{g}_{-1}$ then $A = 0$)
- Jacobi identity, since $\mathfrak{g}$ is fundamental ($\mathfrak{g}_{-1}$ generates $\mathfrak{g}_{-2}$).

TKK construction

Let V be a centerless KTS and define the operators, for $x, y, z \in V$:

$$L_{x,y}(z) := (xyz)\,, \quad \varphi_x(y) := L_{y,x}\,, \quad D_{x,y}(z) := -\varphi_{K_{x,y}(z)}\,.$$

$$\begin{array}{ccccccccccc} \mathfrak{g} = \mathfrak{g}(V) & = & \langle K_{x,y}\rangle & \oplus & V & \oplus & \langle L_{x,y}\rangle & \oplus & \langle \varphi_x\rangle & \oplus & \langle D_{x,y}\rangle \\ & & -2 & & -1 & & 0 & & 1 & & 2 \end{array}$$

Remark: $\mathfrak{g}(V)$ is a subalgebra of the Tanaka prolongation $\mathfrak{g}^\infty$.

Remark: The following map is a grade-reversing involution of $\mathfrak{g}$,

$$\sigma : \qquad K_{x,y} \leftrightarrow D_{x,y}\,, \quad x \leftrightarrow -\varphi_x\,, \quad L_{x,y} \leftrightarrow -L_{y,x}$$

TKK construction

Theorem (C. , – , S.)

There is a one-to-one correspondence between K-simple Kantor triple systems V and pairs $(\mathfrak{g}, \sigma)$, where $\mathfrak{g}$ is a simple 5-graded Lie algebra and σ a grade-reversing involution. Moreover, V is finite dimensional (resp. linearly-compact) if and only if $\mathfrak{g}$ is finite dimensional (resp. linearly-compact).

Infinite-dimensional KTS

Theorem (C. , − , S.)

There are no infinite-dimensional simple linearly-compact KTS.

Example (W(n))

The infinite-dimensional linearly-compact simple Lie algebra of formal vector fields in n indeterminates:

$$W(n) = \left\{ \sum_{i=1}^{n} P_i \frac{\partial}{\partial x_i} \mid P_i \in \mathbb{C}[\![x_1, \ldots, x_n]\!] \right\}$$

Any $\mathbb{Z}$-grading of $W(n)$ is determined by assigning a degree

$$deg(x_i) = k_i \, , \qquad deg(\frac{\partial}{\partial x_i}) = -k_i \, . \tag{4}$$

The grading obtained cannot be a 5-grading ($\mathfrak{g}_i \neq 0$ for infintely-many i).

Finite-dimensional KTS

Problem: How to classify grade-reversing involutions of finite-dimensional simple 5-graded Lie algebras?

Theorem (C. , − , S.)

There is a bijection between isomorphic $(\mathfrak{g}, \sigma)$ *and isomorphic* $(\mathfrak{g}^o, \theta)$*, where* $\mathfrak{g}$ *is a simple complex* $\mathbb{Z}$*-graded Lie algebra,* σ *a grade-reversing involution of* $\mathfrak{g}$*,* $\mathfrak{g}^o$ *is a real absolutely simple* $\mathbb{Z}$*-graded Lie algebra and* θ *a grade-reversing Cartan involution of* $\mathfrak{g}^o$*.*

Solution: Given a 5-graded simple Lie algebra $\mathfrak{g}$, each non-isomorphic KTS structure of $\mathfrak{g}$ corresponds exactly to one of its real forms $\mathfrak{g}^o$.

Derivations of KTS

Theorem (C. , − , S.)

Let V be a K-simple KTS with associated real form $(\mathfrak{g}^o, \theta)$.
Let $(\mathfrak{g}^o_0)^{ss}$ be the semisimple part of $\mathfrak{g}^o_0$ and $(\mathfrak{g}^o_0)^{ss} = \mathfrak{l} \oplus \mathfrak{p}$ its Cartan decomposition with respect to θ.
The Lie algebra of derivations of V, denoted $\mathfrak{der}(V)$, is the complexification of the maximal compact subalgebra $\mathfrak{l}$

$$\mathfrak{der}(V) = \mathfrak{l} \otimes \mathbb{C}$$

Finite-dimensional KTS

Theorem (C. , – , S.)

Up to isomorphisms there are 8 infinite series of classical K-simple KTS and 23 exceptional cases.

The exceptional KTS can be divided into three classes, depending on the grading of the associated Lie algebra:

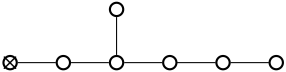
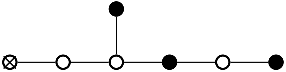
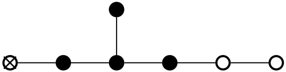
(i) of contact type, if $\dim(\mathfrak{g}_{-2}) = 1$;

(ii) of extended Poincaré type, if $\mathfrak{g}_{-2} = U$ and $\mathfrak{g}_0 \supset \mathfrak{so}(U)$;

(iii) of special type otherwise.

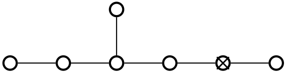
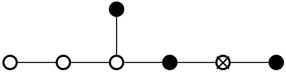
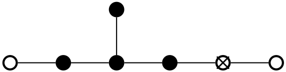
Exceptional KTS associated to E_7

Contact type Poincaré type Special type

$\mathfrak{g}^o$	Satake Diagram	KTS
$E\ V$		$V = \mathbb{S}_6^+$ $\mathfrak{der}(V) = \mathfrak{so}(6,\mathbb{C}) \oplus \mathfrak{so}(6,\mathbb{C})$
$E\ VI$		$V = \mathbb{S}_6^+$ $\mathfrak{der}(V) = \mathfrak{gl}(6,\mathbb{C})$
$E\ VII$		$V = \mathbb{S}_6^+$ $\mathfrak{der}(V) = \mathfrak{so}(2,\mathbb{C}) \oplus \mathfrak{so}(10,\mathbb{C})$

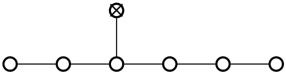
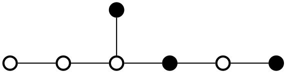
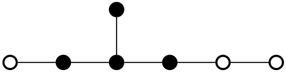
Exceptional KTS associated to E_7

Poincaré type Special type

$\mathfrak{g}^o$	Satake Diagram	KTS
E V		$V = \mathbb{S}_5^+ \otimes \mathbb{C}^2$ $\mathfrak{der}(V) = \mathfrak{so}(5,\mathbb{C}) \oplus \mathfrak{so}(5,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C})$
E VI		$V = \mathbb{S}_5^+ \otimes \mathbb{C}^2$ $\mathfrak{der}(V) = \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(7,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$
E VII		$V = \mathbb{S}_5^+ \otimes \mathbb{C}^2$ $\mathfrak{der}(V) = \mathfrak{so}(9,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C})$

Exceptional KTS associated to E_7

Poincaré type Special type

$\mathfrak{g}^o$	Satake Diagram	KTS
E V		$V = \Lambda^3(\mathbb{C}^7)^*$ $\mathfrak{der}(V) = \mathfrak{so}(7, \mathbb{C})$
E VI		No KTS
E VII		No KTS

Exceptional KTS of special type E_7

$$\mathfrak{g} = \underset{-2}{\Lambda^6(\mathbb{C}^7)^*} \oplus \underset{-1}{\Lambda^3(\mathbb{C}^7)^*} \oplus \underset{0}{\mathfrak{sl}(7,\mathbb{C}) \oplus \mathbb{C}E} \oplus \underset{1}{\Lambda^3\mathbb{C}^7} \oplus \underset{2}{\Lambda^6\mathbb{C}^7}$$

Let η be a scalar product on $\mathbb{C}^7$ and $\sharp$ the associated musical morphism sending $\Lambda^k(\mathbb{C}^7)^* \to \Lambda^k\mathbb{C}^7$.
Let $\bullet$ be the natural projection $\Lambda^3(\mathbb{C}^7)^* \otimes \Lambda^3\mathbb{C}^7 \to \mathfrak{sl}(7,\mathbb{C})$.

Theorem

The vector space $\Lambda^3(\mathbb{C}^7)^*$ with triple product

$$(\alpha\beta\gamma) = \frac{2}{7}\eta(\alpha,\beta)\gamma - (\beta^\sharp \bullet \alpha)\cdot\gamma \tag{5}$$

is a K-simple KTS with associated Lie algebra E_7 and derivation algebra $\mathfrak{so}(7,\mathbb{C})$.

Thanks!