SOME CONJECTURES ON COMPLEX FINITE-DIMENSIONAL SOLVABLE LEIBNIZ ALGEBRAS

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Non-associative Algebras in Cádiz, February 23, 2018

Unión Europea– Fondo Europeo de Desarrollo Regional
Ministerio de Economía, industria y Competitividad
MTM2016-79661-P Agencia Estatal de Investigación

Introduction

A Leibniz algebra is a non-commutative analogue of a Lie algebra, in the sense that adding antisymmetry to the Leibniz bracket leads to coincidence of fundamental identity (the Leibniz identity) with the Jacobi identity. Therefore, a Lie algebra is a particular case of a Leibniz algebra.

Leibniz algebras were introduced by J.-L. Loday in 1993 and since then the study of Leibniz algebras has been carried on intensively.

Investigation of Leibniz algebras shows that classical results on Cartan subalgebras, Levi's decomposition, Engel's and Lie's theorems, properties of solvable algebras with given nilradical and others from theory of Lie algebras have been extended to Leibniz algebras case.

Introduction

Definition

An algebra $(L, [\cdot, \cdot])$ *over a field* $\mathbb{F}$ *is called a* *Leibniz algebra* *if it is defined by the identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \quad \textit{for all} \quad x, y \in L,$$

which is called the *Leibniz identity.*

Introduction

From the Leibniz identity it follows that a Leibniz algebra preserves a unique property of Lie algebras: the operator of right multiplication on an element of an algebra is a derivation.

It is remarkable that the ideal generated by the squares of elements of a non-Lie Leibniz algebra L plays an important role since it determines the (possible) non-Lie nature of L. From the Leibniz identity, this ideal is contained in the right annihilator of the Leibniz algebra.

Introduction

The following theorem recently proved by D. Barnes in [On Levi's theorem for Leibniz algebras. Bull. Australian Math. Soc., 2012] presents an analogue of Levi–Malcev's theorem for Leibniz algebras.

Theorem (Barnes)

Let $\mathcal{L}$ be a finite-dimensional Leibniz algebra over a field of characteristic zero and let $\mathcal{R}$ be its solvable radical. Then there exists a semisimple Lie subalgebra $\mathcal{S}$ of $\mathcal{L}$ such that $\mathcal{L} = \mathcal{S} \dot{+} \mathcal{R}$.

Therefore, the study of finite-dimensional Leibniz algebras is focused to solvable algebras. The procedure how to define a solvable Lie algebra by means of a fixed nilradical was presented by Mubarakzyanov in [On solvable Lie algebras, Izv. Vysv s. Uv cehn. Zaved. Matematika, 1963].

The main tool in the procedure is the using the properties of outer derivations of nilradical. More exactly, outer derivations which are nil-independent. The method of the description of a solvable Leibniz algebra with a given nilradical is an extension of the procedure of Mubarakzyanov.

Solvable Leibniz algebras with a given nilradical

For a given Leibniz algebra L, we define the lower central and derived series to the sequences of two-sided ideals defined recursively as follows:

- $L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1;$
- $L^{[1]} = L, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1.$

Definition

A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$).

The minimal number n with such property is said to be the index of nilpotency of the algebra L.

For a fixed $x \in L$, the map $\mathcal{R}_x \colon L \to L, \mathcal{R}_x(y) = [y, x]$ is a derivation. We call this kind of derivations as inner derivations.
Derivations that are not inner are said to be outer derivations.

Definition

Let $d_1, d_2, \dots, d_n$ be derivations of a Leibniz algebra L. The derivations $d_1, d_2, \dots, d_n$ are said to be nil-independent if

$$\alpha_1 d_1 + \alpha_2 d_2 + \dots + \alpha_n d_n$$

is not nilpotent for any scalars $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$, which are not all zero.

In other words, if for any $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ there exists a natural number k such that $(\alpha_1 d_1 + \alpha_2 d_2 + \dots + \alpha_n d_2)^k = 0$, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Solvable Leibniz algebras with a given nilradical

Let R be a solvable Leibniz algebra. Then it can be decomposed into the form $R = N \oplus Q$, where N is the nilradical and Q is the complementary vector space. Since the square of a solvable algebra is a nilpotent ideal and the finite sum of nilpotent ideals is a nilpotent ideal too, then the ideal R^2 is nilpotent, i.e. $R^2 \subseteq N$ and consequently, $Q^2 \subseteq N$.

Theorem

Let R be a solvable Leibniz algebra and N its nilradical. Then the dimension of the complementary vector space to N is not greater than the maximal number of nil-independent derivations of N.

Solvable Leibniz algebras with a given nilradical

Definition

An n-dimensional Leibniz algebra is said to be null-filiform if
$\dim L^i = n + 1 - i, \ 1 \leq i \leq n + 1.$

It is clear that a null-filiform Leibniz algebra has maximal index of nilpotency.

Sh.A. Ayupov and B.A. Omirov classified null-filiform Leibniz algebras.

Theorem (Ayupov, O.)

An arbitrary n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra:

$$NF_n : \quad [e_i, e_1] = e_{i+1}, \ 1 \leq i \leq n-1,$$

where $\{e_1, e_2, \dots, e_n\}$ *is a basis of the algebra* NF_n*.*

From this theorem it is easy to see that a nilpotent Leibniz algebra is null-filiform if and only if it is a one-generated algebra, i.e. an algebra generated by a simple element.

Note that this notion has no sense in Lie algebras case, because they are at least two-generated.

J.M. Casas, M. Ladra, B.A. Omirov, I.A. Karimjanov in [Classification of solvable Leibniz algebras with null-filiform nilradical. Linear and Multilinear Algebra, 2013] proved that the maximal number of nil-independent derivations of the n-dimensional null-filiform Leibniz algebra NF_n is 1.

Therefore, the dimension of a solvable Leibniz algebra with nilradical NF_n is equal to $n + 1$.

Moreover, we get the description of such algebras.

Solvable Leibniz algebras with null-filiform nilradical

Theorem

Let $R(NF_n)$ be a solvable Leibniz algebra whose nilradical is NF_n. Then there exists a basis $\{e_1, e_2, \dots, e_n, x\}$ of the algebra $R(NF_n)$ such that the multiplication table of the algebra with respect to this basis has the following form:

$$R(NF_n): \quad \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-1, \\ [x, e_1] = -e_1, & \\ [e_i, x] = ie_i, & 1 \le i \le n. \end{cases}$$

Observation

Note that in the table of multiplication of the algebra $R(NF_n)$ the operator R_x has diagonal matrix form and $[e_1, x] = e_1, [x, x] = 0$.

Solvable Lie algebras with naturally graded filiform nilradicals

Now we present the classification of naturally graded filiform Lie algebras obtained by M. Vergne in [*Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes*, Bull. Soc. Math. France, 1970].

Theorem (Vergne)

Any complex naturally graded filiform Lie algebra is isomorphic to one of the following non-isomorphic algebras:

$$L_n : [e_i, e_1] = -[e_1, e_i] = e_{i+1}, \quad 2 \leq i \leq n-1.$$

$$Q_{2n} : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq 2n-2, \\ [e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, & 2 \leq i \leq n, \end{cases}$$

Let us give the classification of solvable Lie algebras with nilradical L_n obtained by L. Šnobl and P. Winternitz in [A class of solvable Lie algebras and their Casimir invariants. J. Phys. A, 2005].

Theorem

There exists only one class of solvable Lie algebras of dimension $n+2$ with nilradical L_n. It is represented by a basis $\{e_1, e_2, \dots, e_n, x, y\}$ and the Lie brackets are

$$R(L_n): \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \le i \le n-1, \\ [e_i, x] = -[x, e_i] = (i-2)\, e_i, & 2 \le i \le n, \\ [e_1, x] = -[x, e_1] = e_1, & \\ [e_i, y] = -[y, e_i] = e_i, & 2 \le i \le n. \end{cases}$$

J. M. Ancochea Bermúdez, et al. [Solvable Lie algebras with naturally graded nilradicals. J. Phys. A, 2006] proved the following result.

Theorem

For any $n \geq 3$ there is only one $(2n+2)$-dimensional solvable Lie algebra having a nilradical isomorphic to Q_{2n}:

$$R(Q_{2n}): \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq 2n-2, \\ [e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i \, e_{2n}, & 2 \leq i \leq n, \\ [e_i, x] = -[x, e_i] = i \, e_i, & 1 \leq i \leq 2n-1, \\ [e_{2n}, x] = -[x, e_{2n}] = (2n+1) \, e_{2n}, & \\ [e_i, y] = -[y, e_i] = e_i, & 1 \leq i \leq 2n-1, \\ [e_{2n}, y] = -[y, e_{2n}] = 2 \, e_{2n}. & \end{cases}$$

Observation

Note that in the tables of multiplications of the algebras $R(L_n)$, $R(Q_{2n})$ the operators of right multiplications R_x, R_y have diagonal matrix form with $[e_1, x] = e_1, [e_2, y] = e_2$ and $[Q, Q] = 0$ for $Q = \langle x, y \rangle$.

Solvable Leibniz algebras with naturally graded non-Lie filiform nilradicals

In the following theorem we recall the classification of the naturally graded filiform non-Lie Leibniz algebras obtained by Sh.A. Ayupov and B.A. Omirov.

Theorem (Ayupov, O.)

Any complex n-dimensional naturally graded filiform non-Lie Leibniz algebra is isomorphic to one of the following non isomorphic algebras:

$$F_n^1 = \begin{cases} [e_1, e_1] & = & e_3, \\ [e_i, e_1] & = & e_{i+1}, \quad 2 \leq i \leq n-1, \end{cases}$$

$$NF_{n-1} \oplus \mathbb{C} \cong F_n^2 = \begin{cases} [e_1, e_1] & = & e_3, \\ [e_i, e_1] & = & e_{i+1}, \quad 3 \leq i \leq n-1. \end{cases}$$

The following two results were obtained by J.M. Casas, M. Ladra, B.A. Omirov and I.A. Karimjanov in [Classification of solvable Leibniz algebras with naturally graded filiform nilradical. Linear Algebra and its Application, 2013].

Theorem

An arbitrary $(n+2)$-dimensional solvable Leibniz algebra with nilradical F_n^1 is isomorphic to the following algebra

$$R(F_n^1):\begin{cases} [e_i,e_1]=e_{i+1}, \quad 2\leq i\leq n-1, & [e_1,x]=e_1,\\ [e_i,y]=e_i, \quad 2\leq i\leq n, & [e_i,x]=(i-1)e_i, \quad 2\leq i\leq n,\\ & [x,e_1]=-e_1. \end{cases}$$

Theorem

An arbitrary $(n+2)$-dimensional solvable Leibniz algebra with nilradical F_n^2 is isomorphic to one of the following non-isomorphic algebras:

$$R(F_n^2)_\alpha : \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 3 \le i \le n-1, \\ [e_1, x] = e_1, & [x, e_1] = -e_1, \\ [e_2, y] = e_2, & [e_i, x] = (i-1)e_i, \quad 3 \le i \le n, \\ [y, e_2] = (\alpha - 1)e_2, & \alpha \in \{0, 1\}. \end{cases}$$

Observation

Note that in the tables of multiplications of the algebras $R(F_n^1)$, $R(F_n^2)_\alpha$ the operators of right multiplications R_x, R_y have diagonal matrix form with $[e_1, x] = e_1, [e_2, y] = e_2$ and $[Q, Q] = 0$ for $Q = \langle x, y \rangle$.

Solvable Leibniz algebras whose nilradical is a direct sum of some ideals

L.M. Camacho, K. Masutova, B.A. Omirov, I.M. Rikhsiboev in [Solvable Leibniz algebras with $N = NF_n \bigoplus F_m^1$ nilradical, Open Math., 2013] proved the following result.

Theorem

In the case when the dimension of the supplementary space dim $Q = 3$ any solvable Leibniz algebra with nilradical $N = NF_n \bigoplus F_m^1$ is isomorphic to the following algebra:

$$R(NF_n \oplus F_m^1): \begin{cases} [e_i, e_1] = e_{i+1},\ 1 \leq i \leq n-1, & [f_i, f_1] = f_{i+1},\ 2 \leq i \leq m-1, \\ [e_i, x] = ie_i,\ 1 \leq i \leq n, & [f_1, y] = f_1, \\ [x, e_1] = -e_1, & [f_i, y] = (i-1)f_i,\ 2 \leq i \leq m, \\ & [f_i, z] = f_i,\ 2 \leq i \leq m, \\ & [y, f_1] = -f_1. \end{cases}$$

Observation

Note that in the table of multiplications of the algebra $R(NF_n \oplus F_m^1)$ *the operators of right multiplications* R_x, R_y, R_z *have diagonal matrix form with* $[e_1, x] = e_1, [f_1, y] = f_1, [f_2, z] = f_2$ *and* $[Q, Q] = 0$ *for* $Q = \langle x, y, z \rangle$.

A.Kh. Khudoyberdiyev, M. Ladra, B.A. Omirov in [On solvable Leibniz algebras whose nilradical is a direct sum of null-filiform algebras. Linear and Multilinear Algebra, 2014] presented the classification of solvable Leibniz algebras whose nilradical is a direct sum of null-filiform ideals.

Theorem

Let $R = N \oplus Q$, where $N = NF_1 \oplus \cdots \oplus NF_s$ and $\dim Q = s$. Then R is isomorphic to the following algebra:

$$R(NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s) : \begin{cases} [e_t^i, e_1^i] = e_{t+1}^i, & 1 \leq i \leq s, \ 1 \leq t \leq n_i - 1 \\ [e_t^i, x_i] = te_t^i, & 1 \leq i \leq s, \ 1 \leq t \leq n_i, \\ [x_i, e_1^i] = -e_1^i, & 1 \leq i \leq s, \ 1 \leq t \leq n_i. \end{cases}$$

Observation

Note that in the table of multiplications of the algebra $R(NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s)$ the operators of right multiplications R_{x_i} have diagonal matrix form with $[e_1^i, x_i] = e_t^1$ for $1 \leq i \leq s$ and $[x_i, x_j] = 0$ for $1 \leq i, j \leq s$.

Recently, A. Hof, J. Shade, W. Whiting have proved the following theorems.

Theorem (I)

Any $(n+3)$-dimensional solvable Leibniz algebra $(n \geq 5)$ with nilradical $L_{n-1} \oplus \mathbb{C}$ is isomorphic to one of the following non-isomorphic algebras:

$$R(L_{n-1} \oplus \mathbb{C})_\alpha : \quad \begin{cases} [e_0, e_i] = -[e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_i, x] = -[x, e_i] = (i+1)e_i, & 0 \leq i \leq n-2, \\ [e_i, y] = -[y, e_i] = e_i, & 1 \leq i \leq n-2, \\ [e_{n-1}, z] = e_{n-1}, & \\ [z, e_{n-1}, z] = (\alpha - 1)e_{n-1}, & \alpha \in \{0, 1\}. \end{cases}$$

Theorem (II)

Any $(n+3)$-dimensional solvable Leibniz algebra $(n \geq 7)$ with nilradical $Q_{n-1} \oplus \mathbb{C}$ is isomorphic to one of the following non-isomorphic algebras:

$$R(Q_{n-1} \oplus \mathbb{C})_{\alpha} : \quad \begin{cases} [e_0, e_i] = -[e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-4, \\ [e_i, e_{n-2-i}] = (-1)^{i-1} e_{n-2}, & 1 \leq i \leq n-3, \\ [e_i, x] = -[x, e_i] = (i+1) e_i, & 0 \leq i \leq n-3, \\ [e_{n-2}, x] = -[x, e_{n-2}] = n e_{n-2}, & \\ [e_i, y] = -[e_i, y] = e_i, & 1 \leq i \leq n-3, \\ [e_{n-2}, y] = -[y, e_{n-2}] = 2 e_{n-2}, & \\ [e_{n-1}, z] = e_{n-1}, & \\ [z, e_{n-1}, z] = (\alpha - 1) e_{n-1}, & \alpha \in \{0, 1\}. \end{cases}$$

Observation

Note that in the table of multiplications of the algebra $R(L_{n-1} \oplus \mathbb{C})_\alpha, \ R(Q_{n-1} \oplus \mathbb{C})_\alpha$ *the operators of right multiplications* R_x, R_y, R_z *have diagonal matrix form with* $[e_0, x_i] = e_0$, $[e_1, y] = e_1$, $[e_{n-1}, z] = e_{n-1}$ *and* $[Q, Q] = 0$ *for* $Q = \langle x, y, z \rangle$.

Moreover, these algebras with $\alpha = 0$ *are Lie algebras.*

Now we present the classification of naturally graded quasi-filiform Lie algebras originally published by J.R. Gómez, A. Jiménez-Merchán in [Naturally graded quasi-filiform Lie algebras, Journal of Algebra, 2002] and later corrected by L. García Vergnolle in [Sur les algèbres de Lie quasi-filiformes admettant un tore de dérivations, Manuscripta Mathematica, 2007].

Theorem

Let $(\mathfrak{g}, [-,-])$ be an n-dimensional non-split naturally graded quasi-filiform split Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}$ is isomorphic to one of the following pairwise non-isomorphic algebras with basis $\{e_0, \dots, e_{n-1}\}$:
$n \geq 5$, r odd and $3 \leq r \leq 2\lfloor \frac{n-1}{2} \rfloor - 1$ odd,

$$\mathfrak{L}_{n,r}: \quad \begin{cases} [e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_i, e_{r-i}] = (-1)^{i-1} e_{n-1}, & 1 \leq i \leq r-1, \end{cases}$$

$n \geq 7$ odd and $3 \leq r \leq n-4$ odd,

$$\mathfrak{Q}_{n,r}: \quad \begin{cases} [e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-4, \\ [e_i, e_{r-i}] = (-1)^{i-1} e_{n-1}, & 1 \leq i \leq r-1, \\ [e_i, e_{n-2-i}] = (-1)^{i-1} e_{n-2}, & 1 \leq i \leq n-3, \end{cases}$$

$n \geq 7$ odd,

$$\mathfrak{T}_{n,n-4}: \quad \begin{cases} [e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-5, \\ [e_0, e_{n-3}] = e_{n-2}, & \\ [e_0, e_{n-1}] = e_{n-3}, & \\ [e_i, e_{n-4-i}] = (-1)^{i-1} e_{n-1}, & 1 \leq i \leq n-5, \\ [e_i, e_{n-3-i}] = (-1)^{i-1} \frac{n-3-2i}{2} e_{n-3}, & 1 \leq i \leq n-4, \\ [e_i, e_{n-2-i}] = (-1)^{i} (i-1) \frac{n-3-i}{2} e_{n-2}, & 2 \leq i \leq n-4 \end{cases}$$

$n \geq 6$ even

$$\mathfrak{T}_{n,n-3}: \quad \begin{cases} [e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-4, \\ [e_0, e_{n-1}] = e_{n-2}, & \\ [e_i, e_{n-3-i}] = (-1)^{i-1} e_{n-1}, & 1 \leq i \leq n-4, \\ [e_i, e_{n-2-i}] = (-1)^{i-1} (\frac{n-2-2i}{2}) e_{n-2}, & 1 \leq i \leq n-3. \end{cases}$$

The following four results are also proved by A. Hof, J. Shade, W. Whiting.

Theorem (1)

Any Leibniz algebra of dimension $n+2$, $n \geq 5$, with nilradical $\mathfrak{L}_{n,r}$ is isomorphic to the Lie algebra with basis $\{e_0, \dots, e_{n-1}, x, y\}$ and the following multiplication table

$$\mathfrak{L}_{n,r}$$

$$\begin{array}{ll} [e_i, x] = (i+1)e_i, & 0 \leq i \leq n-2, \\ [e_i, y] = e_i, & 1 \leq i \leq n-2, \\ [e_{n-1}, x] = (r+2)e_{n-1}, & \\ [e_{n-1}, y] = 2e_{n-1}. & \end{array}$$

Theorem (2)

Any Leibniz algebra of dimension $n+2$, $n \geq 7$ odd and $3 \leq r \leq n-4$ odd, with nilradical $\mathfrak{Q}_{n,r}$, is isomorphic to the Lie algebra with basis $\{e_0, \dots, e_{n-1}\}$ and the following multiplication table

$$\mathfrak{Q}_{n,r}$$

$$\begin{array}{ll} [e_i, x] = (i+1)e_i, & 0 \leq i \leq n-2, \\ [e_{n-1}, x] = (r+4)e_{n-1}, & \\ [e_i, y] = e_i, & 1 \leq i \leq n-3, \\ [e_i, y] = 2e_i, & n-2 \leq i \leq n-1. \end{array}$$

Theorem (3)

Any solvable Leibniz algebra of dimension $n+2$, $n \geq 7$ odd, with nilradical $\mathfrak{T}_{n,n-4}$ is isomorphic to the Lie algebra with basis $\{e_0, \ldots, e_{n-1}, x, y\}$ and the following multiplication table.

$$\mathfrak{T}_{n,n-4}$$

$$\begin{array}{ll}
[e_i, x] = (i+1)e_i, & 0 \leq i \leq n-4, \\
[e_i, y] = e_i, & 1 \leq i \leq n-4, \\
[e_{n-3}, x] = (n-1)e_{n-3}, [e_{n-2}, x] = ne_{n-2}, & \\
[e_{n-1}, x] = (n-2)e_{n-1}, & \\
[e_i, y] = 2e_i, & n-3 \leq i \leq n-1.
\end{array}$$

Theorem (4)

Any solvable Leibniz algebra of dimension $n+2$, $n \geq 6$ even, with nilradical $\mathfrak{T}_{n,n-3}$, is isomorphic to the Lie algebra with basis $\{e_1, \dots, e_n, x, y\}$ and the following non-zero products:

$$\mathfrak{T}_{n,n-3}$$

$$\begin{array}{ll}
[e_i, x] = (i+1)e_i, & 0 \leq i \leq n-3, \\
[e_i, y] = e_i, & 1 \leq i \leq n-3, \\
[e_{n-1}, x] = ne_{n-1}, & \\
[e_n, x] = (n-1)e_n, & \\
[e_i, y] = 2e_i, & i \in \{n-2, n-1\}.
\end{array}$$

Let $R(N) = N \oplus Q$ be a solvable Leibniz algebra with nilradical N and let $\{e_i \mid 1 \leq i \leq k\}$ are generators of N.

Hypothesis 1: Let $R(N) = N \oplus Q$ be a solvable Leibniz algebra with nilradical N and dimension of the complemented space Q is equal to the number of generators of the nilradical N. Then for any basis elements $x_i \in Q, \ 1 \leq i \leq k$ the operators of right multiplications R_{x_i} are diagonal with $[e_i, x_i] = e_i, \ [e_i, x_j] = 0, \ 1 \leq i \neq j \leq k$ and $[Q, Q] = 0$.

Theorem

Let $R = N \oplus Q$ *be a solvable Leibniz algebra such that* $\dim Q = \dim N/N^2 = k$. *Then* R *admits a basis* $\{e_1, e_2, \ldots, e_n, x_1, x_2, \ldots, x_k\}$ *such that the table of multiplication in* R *has the following form:*

$$\begin{cases} [e_i, e_j] = \sum\limits_{t=k+1}^{n} \gamma_{i,j}^t e_t, & 1 \leq i, j \leq n, \\ [e_i, x_i] = e_i, & 1 \leq i \leq k, \\ [x_i, e_i] = (b_i - 1)e_i, & b_i \in \{0, 1\}, 1 \leq i \leq k, \\ [e_i, x_j] = \alpha_{i,j} e_i, & k+1 \leq i \leq n, \ \ 1 \leq j \leq k, \\ [x_j, e_i] = \sum\limits_{t=1}^{q} \beta_{j,i}^{i_t} e_{i_t}, & k+1 \leq i \leq n, \ 1 \leq j \leq k, \end{cases}$$

where $\alpha_{i,j}$ *is the number of entries of a generator basis element* e_j *involved in forming of non generator basis element* e_i.

Corollary

Let $R = N \oplus Q$ be a solvable Lie algebra such that $\dim Q = \dim N/N^2 = k$. *Then R admits a basis* $\{e_1, e_2, \dots, e_n, x_1, x_2, \dots, x_k\}$ *such that the table of multiplications in R has the following form:*

$$\begin{cases} [e_i, e_j] = \sum\limits_{t=k+1}^{n} \gamma_{i,j}^t e_t, & 1 \leq i \neq j \leq n, \\ [e_i, x_i] = -[x_i, e_i] = e_i, & 1 \leq i \leq k, \\ [e_i, x_j] = -[x_j, e_i] = \alpha_{i,j} e_i, & k+1 \leq i \leq n, \quad 1 \leq j \leq k. \end{cases} \tag{1}$$

Below we present a result which asserts that if nilradical is non-split Lie algebra under our condition on the dimension of complementary subspace, then Leibniz algebra becomes to a Lie algebra.

Theorem

Let $R = N \oplus Q$ be a solvable Leibniz algebra such that $\dim Q = \dim N/N^2 = k$ and N be a non-split Lie algebra. Then R is a Lie algebra with the table of multiplications (1).

Let $\{e_1, e_2, \ldots, e_{k_1}\} \cap Ann_r(N) = \emptyset$ and $\{e_{k_1+1}, e_{k_1+2}, \ldots, e_k\} \subset Ann_r(N)$.

Proposition

Let $R = N \oplus Q$ be a solvable Leibniz algebra such that $\dim Q = \dim N/N^2 = k$ and N be a non-split nilradical. Then R is isomorphic to the unique algebra with the following table of multiplication:

$$\begin{cases} [e_i, e_j] = \sum\limits_{t=k+1}^{n} \gamma_{i,j}^t e_t, & 1 \leq i, j \leq n, \\ [e_i, x_i] = e_i, & 1 \leq i \leq k, \\ [x_i, e_i] = -e_i, & 1 \leq i \leq k_1, \\ [e_i, x_j] = \alpha_{i,j} e_i, & k+1 \leq i \leq n, \quad 1 \leq j \leq k, \\ [x_j, e_i] = \sum\limits_{t=1}^{q} \beta_{j,i}^{i_t} e_{i_t}, & k+1 \leq i \leq n, \ 1 \leq j \leq k. \end{cases} \tag{2}$$

Theorem

Let $R = N \oplus Q$ *be a solvable Leibniz algebra with* $\dim Q = \dim N/N^2 = k$ *and* $N = \bigoplus_{t=1}^{s} N_t$, *where* N_t *is non-split non abelian ideal of* N. *Then* $R = \bigoplus_{t=1}^{s} R_t$, *where* $R_t = N_t \oplus Q_t$ *are ideals of* R *with tables of multiplications of the form (2).*

Theorem

Let $R = N + Q$ be a solvable Leibniz algebra such that $\dim Q = \dim N/N^2 = k + p$ *and* $N = \bigoplus_{t=1}^{s} N_t \oplus \mathbb{C}^p$*, where* N_t *is non-split non abelian ideal of* N *with* $\dim N_t = n_t$*. Then* $R = \bigoplus_{t=1}^{s} R_t + R(\mathbb{C}^p)$ *with* $R_t = N_t + Q_t$ *and its table of multiplications has the following form*

$$R(b_1, b_2, \dots .b_p):$$

$$\begin{cases}
[e_i^t, e_j^t] = \sum\limits_{p=k_t+1}^{n_1} \gamma_{i,j}^{t,p} e_t, & 1 \le i,j \le n_t,\ 1 \le t \le s,\\
[e_i^t, x_i^t] = -[x_i^t, e_i^t] = e_i^t, & 1 \le i \le k_t,\ 1 \le t \le s,\\
[e_i^t, x_j^t] = \alpha_{i,j}^t e_i^t, & k_t \le i \le n_t,\ 1 \le j \le k_t,\ 1 \le t \le s,\\
[x_j^t, e_i^t] = \sum\limits_{l=1}^{q} \beta_{t,j,i}^{i_l} e_{i_l}^t, & k_t + 1 \le i \le n_t,\ 1 \le j \le k_t,\ 1 \le t \le s,\\
[f_i, y_i] = f_i, & 1 \le i \le p,\\
[y_i, f_i] = (b_i - 1) f_i, & b_i \in \{0, 1\},\ 1 \le i \le p.
\end{cases}$$

Corollary

Under the conditions of the above theorem up to isomorphism there exist only $p+1$ algebras:

$$R(0,0,\dots,0),\ R(1,0,\dots,0),\ R(1,1,0,\dots,0),\dots,$$

$$R(1,1,\dots,1,0),\quad R(1,1\dots,1).$$

Observation

In the case when the dimension of the complemented space Q is not equal to the number of generators of the nilradical N there exists an example such that $[Q, Q] \neq 0$ and $R_{x_{i_0}}$ is not diagonal for some $1 \leq i_0 < k$.

On triviality of the first group of cohomology of some solvable Leibniz algebras

Theorem

Let $R = N \oplus Q$ be a solvable Lie algebra such that $\dim Q = \dim(N/N^2)$. *Then* $H^1(R, R) = 0$.

Theorem

Let $R = N \oplus Q$ be a solvable Leibniz algebra such that $\dim Q = \dim(N/N^2)$. *Then* $HL^1(R, R) = 0$.

The second group of cohomology of Leibniz algebras

First, we give the definition of the second group of cohomology.
The second cohomology group for a Leibniz algebra L and its representation M is defined as the quotient space

$$HL^2(L, M) = ZL^2(L, M)/BL^2(L, M),$$

where the elements $ZL^2(L, M)$ and $BL^2(L, M)$ are called 2-cocycles and 2-coboundaries, respectively.

The elements $f \in BL^2(L, L)$ and $\varphi \in ZL^2(L, L)$ are defined as follows

$$f(x, y) = [d(x), y] + [x, d(y)] - d([x, y]) \text{ for some linear map } d$$

and

$$(d^2\varphi)(x, y, z) = [x, \varphi(y, z)] - [\varphi(x, y), z] + [\varphi(x, z), y] \\ + \varphi(x, [y, z]) - \varphi([x, y], z) + \varphi([x, z], y) = 0.$$

The linear reductive group $GL_n(F)$ acts on $Leib_n$ via change of basis, i.e.

$$(g * \lambda)(x, y) = g\Big(\lambda\big(g^{-1}(x), g^{-1}(y)\big)\Big), \quad g \in GL_n(F),\ \lambda \in Leib_n.$$

The orbits $Orb(-)$ under this action are the isomorphism classes of algebras. Recall, Leibniz algebras with open orbits are called rigid.

From algebraic geometry it is known that an algebraic variety is a union of irreducible components. The closures of orbits of rigid algebras give irreducible components of the variety. That is why the finding of rigid algebras is crucial problem from the geometrical point of view.

Due to D. Balavoine [Déformations et rigidité géométrique des algèbres de Leibniz, Communications in Algebra, 1996] we can apply the general principles for deformations and rigidity of Leibniz algebras.

Namely, it is proved that nullity of the second cohomology group $HL^2(L, L) = 0$ gives a sufficient condition for rigidity.

In addition, it is established that Leibniz algebras for which every formal deformation is equivalent to a trivial deformation are rigid.

On triviality of the second group of cohomology

The combining results of several papers provides the following theorem.

Theorem

The second groups of cohomologies with coefficients itself for the following algebras:

$$R(NF_n), \quad R(L_n), \quad R(Q_{2n}), \quad R(F_n^1), \quad R(F_n^2)_1, \quad R(F_n^2)_2,$$

$$R(NF_n \oplus F_m^1), \quad R(NF_1 \oplus NF_2 \oplus \cdots \oplus NF_s)$$

are trivial !!!

On triviality of the second group of cohomology

So, we could formulate the next hypothesis.

Hypothesis 2: Let $R(N) = N \oplus Q$ be a solvable Leibniz algebra with nilradical N and the dimension of the complemented space Q is equal to the number of generators of the nilradical N. Then $HL^2(R(N), R(N)) = 0$, that is, they are **rigid !!!**

From Hypothesis 2 the natural question arises: whether all rigid solvable Leibniz algebras satisfy the condition that the dimension of the complemented space to the nilradical is equal to the number of generators of the nilradical?

The answer to this question is negative and it is clarified in the following description theorem given by J.M. Casas, M. Ladra, B.A. Omirov, I.A. Karimjanov in [Classification of solvable Leibniz algebras with naturally graded filiform nilradical, Linear Algebra and its Application, 2013].

Theorem

An $(n+1)$-dimensional solvable Leibniz algebra with nilradical F_n^2 is isomorphic to one of the following pairwise non-isomorphic algebras:

$$R_1 : \begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, \quad 3 \le i \le n-1, \\ [e_1, x] = -e_1, & [e_i, x] = -(i-1)\, e_i, \quad 3 \le i \le n, \\ [x, e_1] = e_1, & [x, x] = e_2, \end{cases}$$

$$R_2(\alpha), \quad R_3, \quad R_4(\alpha), \quad R_5, \quad R_6(\alpha_3, \alpha_4, \ldots, \alpha_n, \lambda, \delta).$$

On triviality of the second group of cohomology of some solvable Leibniz algebras

Proposition

$\dim HL^2(R_1, R_1) = 0$ *and* $\dim HL^2(R_3, R_3) = \dim HL^2(R_5, R_5) = 1.$

In the next theorem we present the classification of quasi-filiform Lie algebras of maximum length.

Theorem

Let L be an n-dimensional $(n > 11)$ quasi-filiform Lie algebra of maximum length. Then the algebra L is isomorphic to one of the following pairwise non-isomorphic algebras:

$$g^1_{(n,1)}: \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \le i \le n-2, \\ [e_i, e_{n-i}] = (-1)^i e_n, & 2 \le i \le \frac{n-1}{2}, \ n \ge 5 \ \textit{ and } \ n \textit{ is odd}; \end{cases}$$

$$g^2_{(n,1)}: \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \le i \le n-2, \\ [e_i, e_n] = e_{i+2}, & 2 \le i \le n-3, \ n \ge 5; \end{cases}$$

$$g^3_{(n,1)}: \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \le i \le n-2, \\ [e_i, e_n] = e_{i+2}, & 2 \le i \le n-3, \\ [e_2, e_i] = e_{i+3}, & 3 \le i \le n-4, \ n \ge 7; \end{cases}$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of the algebra.

Theorem

An arbitrary Leibniz algebra of the family $R(g^i_{(n,1)}, 2)$, $i = 1, 2$, is isomorphic to one of the following Lie algebras:

$$R(g^1_{(n,1)}, 2): \begin{cases} [e_1, x] = -[x, e_1] = e_1, & \\ [e_i, x] = -[x, e_i] = (i-2)e_i, & 2 \leq i \leq n-1, \\ [e_n, x] = -[x, e_n] = (n-4)e_n, & \\ [e_i, y] = -[y, e_i] = e_i, & 2 \leq i \leq n-1, \\ [e_n, y] = -[y, e_n] = 2e_n, & \end{cases}$$

$$R(g^2_{(n,1)}, 2): \begin{cases} [e_1, x] = -[x, e_1] = e_1, & \\ [e_i, x] = -[x, e_i] = (i-2)e_i, & 2 \leq i \leq n-1, \\ [e_n, x] = -[x, e_n] = 2e_n, & \\ [e_i, y] = -[y, e_i] = e_i, & 2 \leq i \leq n-1. \end{cases}$$

Theorem

$$H^2(R(g^1_{(n,1)},2),R(g^1_{(n,1)},2)) = H^2(R(g^2_{(n,1)},2),R(g^2_{(n,1)},2)) = 0.$$

Definition

For n-dimensional nilradical N of a solvable Leibniz algebra $R = N \oplus Q$ with $\dim(N/N^2) = t$ we associate the linear system $S(N)$ of equations n variables $c_1, \dots, c_t, c_{t+1}, \dots, c_n$ consisting of equalities $c_i + c_j = c_k,\ 1 \leq i, j \leq t$, if and only if a vector $[e_i, e_j]$ contains e_k with a non-zero coefficient.

Hypothesis 3: Let $R(N) = N \oplus Q$ be a solvable Leibniz algebra with nilradical N such that the space of fundamental solutions of the system $S(N)$ has dimension equal to $(t-1)$ and the dimension of the complemented space Q is equal to $(t-1)$. Then the table of multiplication of R has the table of multiplication of the form (1) and $HL^2(R(N), R(N)) = 0$.

The most complete description of solvable Lie algebras with various types of nilradicals obtained till the present time is given in the book:

Libor Šnobl and Pavel Winternitz.
CLASSIFICATION AND IDENTIFICATION OF LIE ALGEBRAS.
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