Leibniz algebras constructed by representations of general Diamond Lie algebras

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The First International Workshop Non-associative algebras
February 21 - 24, 2018

Unión Europea– Fondo Europeo de Desarrollo Regional
Ministerio de Economía, industria y Competitividad
MTM2016-79661-P Agencia Estatal de Investigación

Outline

Definition

An algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is called a Lie algebra if its multiplication (denoted by $(x, y) \mapsto [x, y]$) satisfies the identities:
(1) $[x, x] = 0$,
(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$,
for all x, y, z in $\mathfrak{g}$.

Let $\mathfrak{gl}(\mathbb{C}, n)$ be the general linear algebra, formed by all the complex $n \times n$ matrices, for some $n \in \mathbb{N}$.

Given a Lie algebra $\mathfrak{g}$, a representation of $\mathfrak{g}$ in $\mathbb{C}^n$ is a homomorphism of Lie algebras $f\colon \mathfrak{g} \to \mathfrak{gl}(\mathbb{C}^n) = \mathfrak{gl}(\mathbb{C}, n)$. If the homomorphism f is injective then the representation called a faithful representation. The natural integer n is called the dimension of this representation.

Representations can be also defined by using arbitrary n -dimensional vector spaces V. In such a case, a representation would be a homomorphism of Lie algebra from $\mathfrak{g}$ to the Lie algebra $\mathfrak{gl}(V)$ of endomorphisms of the vector space V, which is called $\mathfrak{g}$-module. However, it is sufficient to consider representations on $\mathbb{C}^n$ because there always exists a unique $n \in \mathbb{N}$ such that V is isomorphic to $\mathbb{C}^n$.

The Ado's theorem

Every finite-dimensional Lie algebra $\mathfrak{g}$ over a field K of characteristic zero can be represented as a matrix Lie algebra, formed by square matrices.

However, that result does not specify which is the minimal dimension of the matrices involved in such representations.

Burde, D.
On a refinement of Ado's Theorem.
Arch. Math. (Basel) **70**, 1998, p. 118–127.

$$\mu(\mathfrak{g}) = min\{dim(M) \mid M \text{ a faithful } \mathfrak{g} \text{ - module}\}$$

Let $\mathfrak{g}$ be an abelian Lie algebra of dimension n over an arbitrary field K. Then $\mu(\mathfrak{g}) = [2\sqrt{n-1}]$.

For the Heisenberg Lie algebras $\mu(\mathfrak{h}_m) = m + 1$.

R.Ghanam, I.Strugar, G.Thompson
Matrix representation for low dimensional Lie algebras.
Extracta Mathematica **20(2)**, 2005, p. 151–184.

J. C. Benjumea, J. Nunez and A. F. Tenorio
Minimal linear representations of the low-dimensional nilpotent Lie algebras.
MATH. SCAND. **102**, 2008, p. 17–26.

Dietrich Burde, Bettina Eick, Willem de Graaf.
Computing faithful representations for nilpotent Lie algebras
J. Algebra **322**, 2009, p. 602–612.

Manuel Ceballos, Juan Nunez, Angel F. Tenorio
Representing filiform Lie algebras minimally and faithfully by strictly upper-triangular matrices.
Journal of Algebra and Its Applications **12(4)**, 2013, 15 pages.

General Diamond Lie algebra

The real general Diamond Lie algebra $\mathfrak{D}_m$ is a $(2m+2)$ - dimensional Lie algebra with basis $\{J, P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_m, T\}$ and nonzero relations:

$$[J, P_k] = Q_k, \quad [J, Q_k] = -P_k, \quad [P_k, Q_k] = T, \quad 1 \leq k \leq m.$$

The complexification of the Diamond Lie algebra, $\mathfrak{D}_m(\mathbb{C})$, is $\mathfrak{D}_m \otimes_R \mathbb{C}$, and it the following (complex) basis:

$$P_k^+ = P_k - iQ_k, \quad Q_k^- = P_k + iQ_k, \quad T, \quad J, \quad 1 \leq k \leq m,$$

where i is the imaginary unit, whose nonzero commutators are

$$[J, P_k^+] = iP_k^+, \quad [J, Q_k^-] = -iQ_k^-, \quad [P_k^+, Q_k^-] = 2iT, \ 1 \leq k \leq m.$$

Minimal faithful representation of complex general Diamond Lie algebra

Proposition

Let $\mathfrak{D}_m(\mathbb{C})$ be a $(2m+2)$-dimensional general Diamond Lie algebra with the basis

$$\{J, P_1^+, P_2^+, \dots, P_m^+, Q_1^-, Q_2^-, \dots, Q_m^-, T\}.$$

Then its minimal faithful representation is given by the correspondence

$$\varphi : \theta J + \sum_{k=1}^{m} \alpha_k P_k^+ + \sum_{k=1}^{m} \beta_k Q_k^- + \delta T \longmapsto$$

$$\begin{pmatrix} \frac{im}{m+2}\theta & \alpha_m & \alpha_{m-1} & \dots & \alpha_2 & \alpha_1 & -\frac{i}{2}\delta \\ 0 & -\frac{2i}{m+2}\theta & a_1 & \dots & 0 & 0 & \beta_m \\ 0 & 0 & -\frac{2i}{m+2}\theta & \dots & 0 & 0 & \beta_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{2i}{m+2}\theta & a_1 & \beta_2 \\ 0 & 0 & 0 & \dots & 0 & -\frac{2i}{m+2}\theta & \beta_1 \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{im}{m+2}\theta \end{pmatrix}.$$

Construction of the Module

Let us denote by $V = \mathbb{C}^{m+2}$ the natural $\varphi(\mathfrak{D}_m(\mathbb{C}))$-module and endow it with a $\mathfrak{D}_m(\mathbb{C})$-module structure, $V \times \mathfrak{D}_m(\mathbb{C}) \to V$, given by

$$(x, e) := x\varphi(e),$$

where $x \in V$ and $e \in \mathfrak{D}_m(\mathbb{C})$.

Then the action of $\mathfrak{D}_m(\mathbb{C})$ on $V = \langle X_1, X_2, \ldots, X_{m+2}\rangle$ is given as follows:

$$(*)\begin{cases} (X_1, J) = \frac{im}{m+2}X_1, & \\ (X_k, J) = -\frac{2i}{m+2}X_k, & 2 \le k \le m+1, \\ (X_{m+2}, J) = \frac{im}{m+2}X_{m+2}, & \\ (X_1, P_k^+) = X_{m+2-k}, & 1 \le k \le m, \\ (X_{m+2-k}, Q_k^-) = X_{m+2}, & 1 \le k \le m, \\ (X_1, T) = -\frac{i}{2}X_{m+2}, & \end{cases}$$

and the remaining products in the action being zero.

Faithful representation of real general Diamond Lie algebra

Proposition

Let $\mathfrak{D}_m(\mathbb{R})$ be a $(2m+2)$-dimensional general real Diamond Lie algebra with the basis $\{J, P_1, P_2, \dots, P_m, Q_1, Q_2, \dots, Q_m, T\}$. Then it is isomorphic to a subalgebra of $\mathfrak{sp}(2m+2, \mathbb{R})$ via the map

$$\psi : aJ + \sum_{k=1}^{m} b_k P_k + \sum_{k=1}^{m} c_k Q_k + dT \mapsto$$

$$\left(\begin{array}{ccccc|ccccc}
0 & b_1 & b_2 & \dots & b_m & c_m & \dots & c_2 & c_1 & 2d \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & -a & c_1 \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & -a & 0 & c_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & -a & \dots & 0 & 0 & c_m \\
\hline
0 & 0 & 0 & \dots & a & 0 & \dots & 0 & 0 & -b_m \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & a & \dots & 0 & 0 & \dots & 0 & 0 & -b_2 \\
0 & a & 0 & \dots & 0 & 0 & \dots & 0 & 0 & -b_1 \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0
\end{array}\right).$$

$\mathfrak{D}_m$-module

Let be the action of $\mathfrak{D}_m$ on $V = \langle X_1, X_2, \dots, X_{m+2}\rangle$,
$V \times \mathfrak{D}_m \to V$, given by

$$(**)\begin{cases} (X_k, J) = -X_{2m+3-k}, & 2 \le k \le m+1, \\ (X_k, J) = X_{2m+3-k}, & m+2 \le k \le 2m+1, \\ (X_1, P_k) = X_{k+1}, & 1 \le k \le m, \\ (X_{2m+2-k}, P_k) = -X_{2m+2}, & 1 \le k \le m, \\ (X_1, Q_k) = X_{2m+2-k}, & 1 \le k \le m, \\ (X_{k+1}, Q_k) = X_{2m+2}, & 1 \le k \le m, \\ (X_1, T) = 2X_{2m+2}, & \end{cases}$$

and the remaining products in the action being zero.

Definition

An algebra L over a field $\mathbb{K}$ is called a Leibniz algebra if for any $x, y, z \in L$, the Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

is satisfied, where $[-, -]$ is the multiplication in L.

One of the approaches to the investigation of Leibniz algebras is a description of such algebras whose quotient algebra with respect to the ideal I is a given Lie algebra.

Every non-Lie Leibniz algebra L contains a non-trivial ideal (denoted by I), which is the subspace spanned by the squares of elements of the algebra L.

For a Leibniz algebra, L we consider the homomorphism into the quotient Lie algebra L/I which is called liezation of L.

The map $I \times L/I \to I$ defined as $(v, \bar{x}) \mapsto [v, x], v \in I, x \in L$ endows I with a structure of L/I-module.

Denote by $Q(L) = L/I \oplus I$, then the operation $(-, -)$ defines Leibniz algebra structure on $Q(L)$, where

$$(\overline{x} + v, \overline{y} + w) := [\overline{x}, \overline{y}] + [v, y], \quad (\overline{x}, \overline{y}) = \overline{[x, y]},$$

$$(v, \overline{x}) = [v, x], \quad (\overline{x}, v) = 0, \quad (v, w) = 0, \quad x, y \in L, \; v, w \in I.$$

In fact, this structure of Leibniz algebra is isomorphic to the initial one of L.

Therefore, for a given Lie algebra G and a right G-module M, we can construct a Leibniz algebra $L = G \oplus M$ by the above construction.

Classification of Leibniz algebras

Let L Leibniz algebras such that $L/I \cong \mathfrak{D}_m(\mathbb{C})$ and the I is $\mathfrak{D}_m(\mathbb{C})$-module defined by $(*)$. The action $I \times \mathfrak{D}_m(\mathbb{C}) \to I$ gives rise to a minimal faithful representation of $\mathfrak{D}_m(\mathbb{C})$.

Theorem (One of the main results)

Let L be an arbitrary Leibniz algebra with corresponding Lie algebra $\mathfrak{D}_m(\mathbb{C})$ and the ideal I associated as $\mathfrak{D}_m(\mathbb{C})$-module defined by $(*)$. Then there exists a basis

$$\{J, P_1^+, P_2^+, \dots, P_m^+, Q_1^-, Q_2^-, \dots, Q_m^-, T, X_1, X_2, \dots, X_{m+2}\}$$

of L such that

$$\begin{cases} [J, P_k^+] = iP_k^+, \\ [J, Q_k^-] = -iQ_k^-, \\ [P_k^+, Q_k^-] = 2iT, \\ [X_1, J] = \frac{im}{m+2} X_1, \\ [X_k, J] = -\frac{2i}{m+2} X_k, & 2 \leq k \leq m+1, \\ [X_{m+2}, J] = \frac{im}{m+2} X_{m+2}, \\ [X_1, P_k^+] = X_{m+2-k}, & 1 \leq k \leq m, \\ [X_1, T] = -\frac{i}{2} X_{m+2}, \\ [X_{m+2-k}, Q_k^-] = X_{m+2}, & 1 \leq k \leq m, \end{cases}$$

where the omitted products are equal to zero.

Theorem

An arbitrary real Leibniz algebra with corresponding Lie algebra $\mathfrak{D}_m$, and the I ideal associated as $\mathfrak{D}_m$-module defined by $(**)$, admits a basis $\{J, P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_m, T, X_1, X_2, \ldots, X_{2m+2}\}$ such that the multiplication table $[\mathfrak{D}_m, \mathfrak{D}_m]$ has the following form:

$$
\begin{cases}
[J,J]=a_1X_{2m+2}, & [J,P_k]=-[P_k,J]=Q_k,\\
[J,Q_k]=-[Q_k,J]=-P_k, & [P_k,Q_k]=-[Q_k,P_k]=T,\\
[P_k,P_s]=[Q_k,Q_s]=b_{k,s}X_{2m+2}, & \\
[P_k,Q_s]=[Q_k,P_s]=c_{k,s}X_{2m+2}, & \\
[X_l,J]=-X_{2m+3-l}, & 2\le l\le m+1,\\
[X_l,J]=X_{2m+3-l}, & m+2\le l\le 2m+1,\\
[X_1,P_l]=X_{l+1}, & 1\le l\le m,\\
[X_{2m+2-l},P_k]=-X_{2m+2}, & 1\le l\le m,\\
[X_1,Q_l]=X_{2m+2-l}, & 1\le l\le m,\\
[X_{l+1},Q_l]=X_{2m+2}, & 1\le l\le m,\\
[X_1,T]=2X_{2m+2}, &
\end{cases}
$$

with the restrictions

$$b_{k,s}=-b_{s,k},\qquad c_{k,s}=c_{s,k},\quad 1\le k,s\le m,\ k\ne s.$$

THANKS FOR YOUR ATTENTION!