# Augmented ternary maps and their applications

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# Introduction and preliminary definitions

***Motivation:***

*When we have a ternary structure with some type of grading a complete knowledge of the grading set imply some structural theorems of the graded ternary structure.*

# Introduction and preliminary definitions

***Purpose and scope:***

1. Consider an arbitrary non-empty set.
2. Endow it with a new structure called *$f$-triple*.
3. Study it so as to obtain several decomposition results.
4. Apply the results to the grading set (*considered as an adequate $f$-triple*) of some ternary structures to obtain structural theorems.

# Introduction and preliminary definitions

**Definition**

Let $\mathcal{A}$ be a non-empty set and $\epsilon \notin \mathcal{A}$ an external element to $\mathcal{A}$. Any map

$$f : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A} \cup \{\epsilon\}$$

is called an *augmented ternary map* on $\mathcal{A}$. We will also say that $\mathcal{A}$ is an $f$*-triple*.

# Introduction and preliminary definitions

## Definition

Let $\mathcal{A}$ be an $f$-triple.

1. A subset $S$ of $\mathcal{A}$ is called a *subtriple* of $\mathcal{A}$ if for any $x, y, z \in S$ we have that $f(x, y, z) \in S \cup \{\epsilon\}$.
2. A subtriple $\mathcal{I}$ of $\mathcal{A}$ is called an *ideal* if
$$f(\mathcal{I}, \mathcal{A}, \mathcal{A}) \cup f(\mathcal{A}, \mathcal{I}, \mathcal{A}) \cup f(\mathcal{A}, \mathcal{A}, \mathcal{I}) \subset \mathcal{I} \cup \{\epsilon\}.$$
3. We say that $\mathcal{A}$ is *simple* if its only ideals are $\emptyset$ and $\mathcal{A}$.

# Introduction and preliminary definitions

## Definition

Let $f : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A} \cup \{\epsilon\}$ be an augmented ternary map and $\{B_i : i \in I\}$ a family of pairwise disjoint subsets of $\mathcal{A}$ such that

$$\mathcal{A} = \dot{\bigcup_{i \in I}} B_i.$$

Then we say that $\mathcal{A}$ is the *orthogonal union* of the $\{B_i : i \in I\}$ if for any $i, j \in I$, with $i \neq j$, we have

$$f(\mathcal{A}, B_i, B_j) = f(B_i, \mathcal{A}, B_j) = f(B_i, B_j, \mathcal{A}) = \{\epsilon\}.$$

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# The structure of the $f$-triples

To study the structure of the $f$-triples we introduce an equivalence relation on it. Then, we need to:

1. Introduce a new variable $\overline{x} \notin \mathcal{A}$ for each $x \in \mathcal{A}$, being:

$$\overline{\mathcal{A}} := \{\overline{x} : x \in \mathcal{A}\}.$$

2. Define a map

$$\phi : \mathcal{P}(\mathcal{A}) \times (\mathcal{A} \,\dot{\cup}\, \overline{\mathcal{A}}) \times (\mathcal{A} \,\dot{\cup}\, \overline{\mathcal{A}}) \to \mathcal{P}(\mathcal{A}).$$

# The structure of the $f$-triples

Given any $\sigma \in S_3$ we define

$$u_\sigma : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{P}(\mathcal{A}) \;\text{ and }\; v_\sigma : \mathcal{A} \times \overline{\mathcal{A}} \times \overline{\mathcal{A}} \to \mathcal{P}(\mathcal{A})$$

as:

$$u_\sigma(x_1, x_2, x_3) = \begin{cases} \emptyset & \text{if} \quad f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \epsilon \\ \{y\} & \text{if} \quad f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = y \in \mathcal{A} \end{cases}$$

and

$$v_\sigma(x, \overline{y}_2, \overline{y}_3) = \{y_1 \in \mathcal{A} : x \in u_\sigma(y_1, y_2, y_3)\}.$$

# The structure of the $f$-triples

Now we consider the following operation

$$\mu : \mathcal{A} \times (\mathcal{A} \,\dot\cup\, \overline{\mathcal{A}}) \times (\mathcal{A} \,\dot\cup\, \overline{\mathcal{A}}) \to \mathcal{P}(\mathcal{A})$$

given by:

- $\mu(x, y, z) := \bigcup_{\sigma \in S_3} u_\sigma(x, y, z)$ for any $x, y, z \in \mathcal{A}$.
- $\mu(x, \overline{y}, \overline{z}) := \bigcup_{\sigma \in S_3} v_\sigma(x, \overline{y}, \overline{z})$ for any $x \in \mathcal{A}$ and any $\overline{y}, \overline{z} \in \overline{\mathcal{A}}$.
- $\mu(x, \mathcal{A}, \overline{\mathcal{A}}) = \mu(x, \overline{\mathcal{A}}, \mathcal{A}) = \emptyset$.

# The structure of the $f$-triples

Finally, we get the mapping

$$\phi : \mathcal{P}(\mathcal{A}) \times (\mathcal{A} \,\dot{\cup}\, \overline{\mathcal{A}}) \times (\mathcal{A} \,\dot{\cup}\, \overline{\mathcal{A}}) \to \mathcal{P}(\mathcal{A})$$

which will be defined as

$$\phi(U, y, z) := \bigcup_{x \in U} \mu(x, y, z).$$

# The structure of the $f$-triples

## Definition

For any $x, y \in \mathcal{A}$, we say that $x$ is *connected* to $y$ if

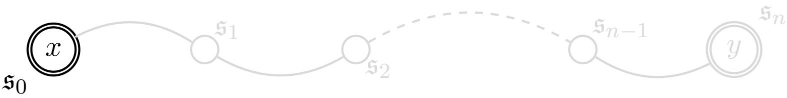
- either $x = y$ or
- there exists $\{a_1, a_2, \ldots, a_{2n}\} \subset \mathcal{A} \dot{\cup} \overline{\mathcal{A}}$, $n \geq 1$, such that

$$\begin{array}{ccl} \emptyset & \neq & \phi(\{x\}, a_1, a_2) \\ \emptyset & \neq & \phi(\phi(\{x\}, a_1, a_2), a_3, a_4) \\ & \vdots & \\ \emptyset & \neq & \phi(\phi(\ldots \phi(\{x\}, a_1, a_2) \ldots ), a_{2n-3}, a_{2n-2}) \\ y & \in & \phi(\phi(\ldots \phi(\{x\}, a_1, a_2) \ldots ), a_{2n-1}, a_{2n}). \end{array}$$

In this case, the subset $\{a_1, \ldots, a_{2n}\}$ is called a *connection* from $x$ to $y$.

# The structure of the $f$-triples

Connection $\{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$.




$$\mathfrak{s}_0 = \{x\}$$

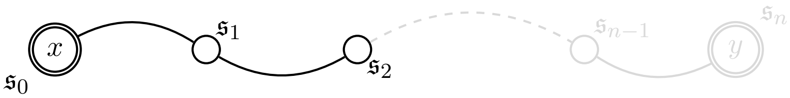
# The structure of the $f$-triples

Connection $\{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$.


$$\mathfrak{s}_1 = \phi(\mathfrak{s}_0, a_1, a_2) \neq \emptyset$$

# The structure of the $f$-triples

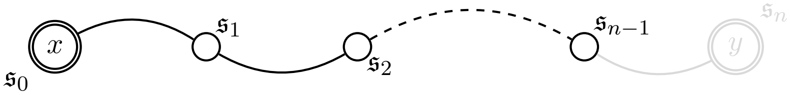
Connection $\{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$.




$$\mathfrak{s}_2 = \phi(\mathfrak{s}_1, a_3, a_4) \neq \emptyset$$

# The structure of the $f$-triples

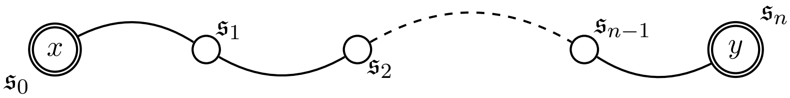
Connection $\{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$.




$$\mathfrak{s}_{n-1} = \phi(\mathfrak{s}_{n-2}, a_{2n-3}, a_{2n-2}) \neq \emptyset$$

# The structure of the $f$-triples

Connection $\{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$.




$$\mathfrak{s}_n = \phi(\mathfrak{s}_{n-1}, a_{2n-1}, a_{2n}) \neq \emptyset$$

# The structure of the $f$-triples

## Proposition

*The relation $\sim$ in $\mathcal{A}$, defined by $x \sim y$ if and only if $x$ is connected to $y$, is an equivalence relation.*

## Theorem

*Let $\mathcal{A}$ be an $f$-triple. Then $\mathcal{A}$ is the orthogonal (disjoint) union*

$$\mathcal{A} = \dot{\bigcup_{[a]\in\mathcal{A}/\sim}} [a]$$

*of the family of the ideals $\{[a] : [a] \in \mathcal{A}/\sim\}$.*

# The structure of the $f$-triples

## Definition

We say that an $f$-triple $\mathcal{A}$ is a *division $f$-triple* if for any $x, i, j, k \in \mathcal{A}$ such that $f(i, j, k) = x$ then

- $i \in f(x, \mathcal{A}, \mathcal{A})$
- $j \in f(\mathcal{A}, x, \mathcal{A})$
- $k \in f(\mathcal{A}, \mathcal{A}, x)$.

# The structure of the $f$-triples

**Theorem**

*Let $\mathcal{A}$ be a division $f$-triple. Then $\mathcal{A}$ is simple if and only if $\mathcal{A}$ has all of its elements connected.*

**Theorem**

*Let $\mathcal{A}$ be a division $f$-triple. Then*

$$\mathcal{A} = \dot{\bigcup_{i \in I}} \mathcal{I}_i$$

*is the orthogonal (disjoint) union of the family $\{\mathcal{I}_i\}_{i \in I}$ of all its nonempty simple ideals.*

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# Applications

## Definition

Let $(\mathcal{T}, \langle \cdot, \cdot, \cdot \rangle)$ be a triple system and $I$ a (non-empty) set. It is said that $\mathcal{T}$ is *graded* by $I$, if

$$\mathcal{T} = \bigoplus_{i \in I} \mathcal{T}_i$$

where any $\mathcal{T}_i$ is a linear subspace of $\mathcal{T}$ satisfying that for any $i, j, k \in I$ either $\langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle = \{0\}$ or $\{0\} \neq \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle \subset \mathcal{T}_r$ for some (unique) $r \in I$. We call *support* of the grading to the set

$$\Sigma := \{i \in I : \mathcal{T}_i \neq \{0\}\}.$$

# Applications

Considering the support $\Sigma$ of the grading and some $\epsilon \notin \Sigma$ we introduce the augmented ternary map $f : \Sigma \times \Sigma \times \Sigma \to \Sigma \cup \{\epsilon\}$ defined by

$$f(i,j,k) := \begin{cases} \epsilon & \text{if} \quad \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle = \{0\} \\ r & \text{if} \quad \{0\} \neq \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle \subset \mathcal{T}_r. \end{cases}$$

Then, for any $[i] \in \Sigma/\sim$ we can define the linear subspace of $\mathcal{T}$

$$\mathcal{T}_{[i]} := \bigoplus_{k \in [i]} \mathcal{T}_k,$$

# Applications

**Definition**

A *homogeneous-ideal* of $\mathcal{T} = \bigoplus_{i \in I} \mathcal{T}_i$ is a linear subspace $\mathcal{I}$ of the form $\mathcal{I} = \bigoplus_{j \in J} \mathcal{T}_j$ with $J \subset I$, and satisfying

$$\langle \mathcal{I}, \mathcal{T}, \mathcal{T} \rangle + \langle \mathcal{T}, \mathcal{I}, \mathcal{T} \rangle + \langle \mathcal{T}, \mathcal{T}, \mathcal{I} \rangle \subset \mathcal{I}.$$

**Definition**

A graded triple system $\mathcal{T}$ is called *homogeneous-simple* if its only homogeneous-ideals are $\{0\}$ and $\mathcal{T}$.

# Applications

**Theorem**

*Let $\mathfrak{T}$ be a triple system graded by a set $I$. Then there is a decomposition of $\mathfrak{T}$ as the direct sum*

$$\mathfrak{T} = \bigoplus_{[i]\in\Sigma/\sim} \mathfrak{T}_{[i]},$$

*being any $\mathfrak{T}_{[i]}$ a nonzero homogeneous-ideal of $\mathfrak{T}$.*

# Applications

## Definition

The grading

$$\mathcal{T} = \bigoplus_{i \in I} \mathcal{T}_i$$

is called a *weak-division* grading if for any $i, j, k \in I$ such that

$$\{0\} \neq \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle \subset \mathcal{T}_r,$$

there exist $x_1, x_2 \in \Sigma$ for any $x \in \{i, j, k\}$ verifying

- $\{0\} \neq \langle \mathcal{T}_r, \mathcal{T}_{i_1}, \mathcal{T}_{i_2} \rangle \subset \mathcal{T}_i$
- $\{0\} \neq \langle \mathcal{T}_{j_1}, \mathcal{T}_r, \mathcal{T}_{j_2} \rangle \subset \mathcal{T}_j$
- $\{0\} \neq \langle \mathcal{T}_{j_1}, \mathcal{T}_{j_2}, \mathcal{T}_r \rangle \subset \mathcal{T}_k$

# Applications

## Proposition

*Let $\mathfrak{T}$ be a triple system with a weak-division grading. Then the $f$-triple $\Sigma$ is a division $f$-triple.*

## Theorem

*Let $\mathfrak{T}$ be a triple system with a weak-division grading. Then*

$$\mathfrak{T} = \bigoplus_{\alpha\in\Omega} \mathfrak{T}_\alpha$$

*is the direct sum of the family $\{\mathfrak{T}_\alpha\}_{\alpha\in\Omega}$ of all its nonzero homogeneous-simple homogeneous-ideals.*

# Applications

Similar structure theorems are given, **also as consequence of the results on augmented ternary maps**, for the classes of

1. Arbitrary supertriple systems with a multiplicative basis
2. Set-graded arbitrary algebraic pairs

Thank you