Quadratic 2-step Lie algebras: Computational algorithms and classification

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Definitions

Definition (Quadratic)

A nonassociative algebra A over a field $\mathbb{K}$ that carry a symmetric and nondegenerate invariant bilinear form f is called **quadratic algebra**.

Where invariant means $f(ab, c) = f(a, bc)$.

Definition (2-step)

A Lie algebra $(\mathfrak{g}, [x, y])$ is 2-step in case $\mathfrak{g}^3 = 0 \neq \mathfrak{g}^2$.

Where $\mathfrak{g}^3 = \operatorname{span}\langle [[xy]z] : x, y, z \in \mathfrak{g} \rangle$ and $\mathfrak{g}^2 = \operatorname{span}\langle [xy] : x, y \in \mathfrak{g} \rangle$.

Definition (Type)

A Lie algebra $(\mathfrak{g}, [x, y])$ has type d when $d = \dim \mathfrak{g} - \dim \mathfrak{g}^2$.

Lemma

The type of any 2-step nilpotent Lie algebra is ≥ 2.

Ussage of quadratic algebras

Example

Any full matrix algebra with its trace.

Example

Any semisimple Lie algebra with its killing form.

Quadratic algebras appears in:

- Cartan Criterion for the semisimplicity of a f.d. Lie algebra.
- Structure theory of naturally reductive homogeneous spaces.
- Theory of modular representations of finite groups.
- Completely integrable Hamiltonian systems
- ...

Existence and importance

In 1997, Noui and Revoy proved that the classification of these type of Lie algebras is **equivalent** to the classification of alternating trilinear forms, an open problem.

Ovando (2007) showed the **existence** of real quadratic 2-step Lie algebras of arbitrary type $d \geq 3$, with $d = 4$ as the only exception. Achieving them as homomorphisms of vectors space with inner products.

These results and the ideas in Benito, de-la-Concepción and Laliena (2017) gives results for the existence and isomorphisms of these algebras over arbitrary fields of characteristic 0.

Cronology

1957-Tsou and Walker: preliminary results on the existence of regular quadratic Lie algebras over the real field.

1985-Medina and Revoy: Introduction to the method of double extension.

1987-Favre and Santharoubane: (characteristic 0) Classification of indecomposable quadratic nilpotent Lie algebras of dimension ≤ 7 (up to isometries, there are exactly 4 algebras)

1997-Noui and Revoy: (Characteristic 0), 2-step quadratic and alternating trilinear forms.

1999-Bordemann: (Characteristic $\neq 2$), Introduction of the method of T^*-extension.

Cronology

2000 to 2014-Several authors:

- Classification of quadratic nilpotent of dimension ≤ 10 over the real field (2007-2014),
- classification of quadratic solvable of dimension ≤ 8 (2014) over algebraically closed fields of characteristic 0,
- irreducible quadratic nonsolvable of dimension ≤ 13 over the complex field (2013).

2007-Ovando: 2-step quadratic real Lie algebras, existence and applications.

2017-Benito, de-la-Concepción and Laliena: (Characteristic 0), Free nilpotent and quadratic nilpotent Lie algebras, a categorical approach.

General tools

- **Basic definition**, structure constants and Jacobi identity.
  Problem: Unfeasible in high dimensions.

- **Double extensions process**: Inductive classification. In each step we add a nilpotent quadratic Lie algebra and a skew symmetric derivation.
  Problem: Determining skew derivations.

- ***T^**-extensions**: It involves the third scalar cohomology $H^3(A, \mathbb{K})$.
  Problem: Very restrictive tool.

- **Free nilpotent Lie algebras and invariant forms**: Quite natural. Lie algebras are provided as **quotients** of free nilpotent Lie algebras endowed with invariant **bilinear forms**. The natural action of the automorphism group of free nilpotent Lie algebras provides the isometrically isomorphic algebras.
  Problem: Determining invariant forms in general.

Quotients of free t-nilpotent Lie algebras

Let $\mathfrak{FL}(\mathfrak{m})$ be the free Lie algebra on the set of generators $\mathfrak{m} = \{x_1, \dots, x_d\}$, $d \geq 2$ we have:

$$\mathfrak{n}_{d,t} = \frac{\mathfrak{FL}(\mathfrak{m})}{\mathfrak{FL}(\mathfrak{m})^{t+1}}.$$

Main features:

- Any t-nilpotent and d-Type is a homomorphic image of $\mathfrak{n}_{d,t}$.
- $\mathrm{Der}\, \mathfrak{n}_{d,t} = \mathfrak{sl}_d(k) \oplus \mathfrak{r}$, $\mathfrak{r}$ solvable radical.
- $\mathrm{Aut}\, \mathfrak{n}_{d,t} \cong GL_d(k) \ltimes N$ (semidirect product).

Free nilpotent vs. quadratics

Lemma (Benito, de-la-Concepción & Lalinea, 2017)

Let $\mathfrak{n}$ the factor Lie algebra $\frac{\mathfrak{n}_{d,t}}{I}$ where I is an ideal such that $\mathfrak{n}_{d,t}^t \not\subseteq I \subseteq \mathfrak{n}_{d,t}^2$. The there exists a symmetric, invariant and non-degenerate bilinear form on $\mathfrak{n}$ if and only if there exist that same form on $\mathfrak{n}_{d,t}$ such that $\mathfrak{n}_{d,t}^\perp = I$.

- $\mathrm{Sym}_0(d,t)$ is the category whose objects are the symmetric invariant bilinear forms ψ on the free Lie algebra $\mathfrak{n}_{d,t}$ s. t. $\operatorname{Ker}\psi \subseteq \mathfrak{n}_{d,t}^2$ and $\mathfrak{n}_{d,t}^t \not\subseteq \operatorname{Ker}\psi$. The morphisms are isometric Lie homomorphisms of $\mathfrak{n}_{d,t}$ up to an equivalence relation.
- $\mathrm{NilpQuad}_{d,t}$ stands for the category whose objects are the t-nilpotent quadratic Lie algebras (n,φ) of type d. The morphisms are Lie homomorphisms that are isometries.
- $Q_{d,t}\colon \mathrm{Sym}_0(d,t) \to \mathrm{NilpQuad}_{d,t}$ the functor

$$Q_{d,t}(\psi) = (\mathfrak{n}_{d,t}/\operatorname{Ker}\psi, \varphi), \quad \varphi(\overline{a},\overline{b}) = \psi(a,b)$$

Equivalence result

Theorem (Benito, de-la-Concepción & Lalinea,2017)

For all $\psi_1, \psi_2 \in \mathsf{Obj}(\mathsf{Sym}_0(d,t))$*, the following assertions are equivalent:*

1. ψ_1 *and* ψ_2 *are isomorphic in* $\mathsf{Sym}_0(d,t)$*.*
2. $Q_{d,t}(\psi_1)$ *and* $Q_{d,t}(\psi_2)$ *are isometrically isomorphic Lie algebras.*
3. *There exists a isometric automorphism* $\theta\colon (\mathfrak{n}_{d,t}, \psi_1) \to (\mathfrak{n}_{d,t}, \psi_2)$*.*

Moreover, the set $Orb_{\mathrm{Aut}\,\mathfrak{n}_{d,t}}(\psi) = \{\psi_\theta : \theta \in \mathrm{Aut}\,\mathfrak{n}_{d,t}\}$ is equal to the set of bilinear invariant symmetric forms isomorphic to ψ in the category $\mathsf{Sym}_0(d,t)$. Therefore the number of orbits of the natural action of $\mathrm{Aut}\,\mathfrak{n}_{d,t}$ is exactly the number of isomorphism types in the classification of t-nilpotent quadratic Lie algebras of type d up to isometries.

Reduction

Any quadratic Lie algebra $(\mathfrak{g}, \varphi)$ satisfies:

- $Z(\mathfrak{g})^{\perp} = \mathfrak{g}^2$ y $Z(\mathfrak{g}) = (\mathfrak{g}^2)^{\perp}$. So, $\dim \mathfrak{g} = \dim \mathfrak{g}^2 + \dim Z(\mathfrak{g})$.
- The dimension of $Z(\mathfrak{g}) \cap \mathfrak{g}^2$ is said isotropic index of $\mathfrak{g}^2$.
- $\mathfrak{g}$ is **reduced** in case $Z(\mathfrak{g}) \subseteq \mathfrak{g}^2$ and the pair (r, s) where $r = \dim \mathfrak{g}^2$ y $s = \dim Z(\mathfrak{g})$ is called the (bi-)type of $\mathfrak{g}$.

Theorem (Tsou-Walker, 1956)

Any non reduced and non abelian quadratic Lie algebra $(\mathfrak{g}, \varphi)$ decomposes as a orthogonal direct sum of proper ideals, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{a}$, where $(\mathfrak{g}_1, \varphi)$ is a reduced and quadratic Lie algebra and $(\mathfrak{a}, \varphi)$ is abelian and quadratic.

Universal example

Let $(\mathfrak{v}, \langle\cdot,\cdot\rangle, \rho)$ be the a triple where $\langle\cdot,\cdot\rangle$ is a nondegenerate bilinear form and $\rho : \mathfrak{v} \to \mathfrak{so}(\mathfrak{v})$ an injective linear map satisfying the identity:

$$\rho(v)(u) + \rho(u)(v) = 0.$$

In the vector space $\mathfrak{n}(\mathfrak{v}, \rho) = \mathfrak{v} \oplus \mathfrak{v}^*$, we introduce:

- the canonical hyperbolic symmetric form $\varphi_{\mathfrak{v}}$ [1] and
- the skew symmetric product $[u + f, v + g] = f_{u,v}$, where $f_{u,v}(w) = \langle\rho(w)(u), v\rangle = -\langle\rho(u)(w), v\rangle$.

Then, $(\mathfrak{n}(\mathfrak{v}, \rho), \varphi_{\mathfrak{v}})$ is a 2-step quadratic Lie algebra.

[1] The hyperbolic symmetric form on a vector space $\mathfrak{v}$ is the form on $\mathbb{H}(\mathfrak{v}) = \mathfrak{v} \oplus \mathfrak{v}^*$, $\mathfrak{v}^*$ dual space, defined as: $\varphi_{\mathfrak{v}}(v_1 + f_1, v_2 + f_2) = f_1(v_2) + f_2(v_1)$.

Theorem (Ovando, 2007)

Up to isometries, the real 2-step quadratic Lie algebras are orthogonal direct sums, as ideals, of the form:

$$(\mathfrak{n}(\mathfrak{v},\rho),\varphi_{\mathfrak{v}})\oplus(\mathbb{R}^m,\langle\cdot,\cdot\rangle),$$

where $(\mathbb{R}^m,\langle\cdot,\cdot\rangle)$ *is the abelian* m*-dimensional Lie algebra endowed with the standard inner product and* $(\mathfrak{n}(\mathfrak{v},\rho),\varphi_{\mathfrak{v}})$ *is as defined in the previous example. Moreover for each* $3\geq d=\dim\mathfrak{v}\neq 4$*, there exists a Lie algebra* $((\mathfrak{v},\rho),\varphi_{\mathfrak{v}})$.

Is it possible to get an algorithmic construction?

Construction

Definition

For any $d \geq 2$, a family $\{A_1, \ldots, A_d\}$ of matrices of order $d \times d$ is called ***d*-quadratic** if the following properties are satisfied:

1. The matrices are skewsymmetric ($A_i^t = -A_i$).
2. The i-th column of every A_i is null.
3. For any $j > i$, the j-th column of A_i is the additive inverse of the i-th column of A_j.
4. If $B_{i<j}$ denotes the submatrix of A_i given by the set of all j-th columns of A_i such that $i < j$, the matrix of order $d \times \frac{d(d-1)}{2}$,

$$B(A_1, \ldots, A_d) = [B_{1<j} B_{2<j} \ldots B_{d-1<j}],$$

has maximal rank d.

Universal construction: algebras

Let $\{A_1, \ldots, A_d\}$ be a d-quadratic family of matrices of order $d \times d$ over any field of characteristic 0. On the $2d$-dimensional vector space $\mathfrak{n}(A_1, \ldots, A_d) = \langle v_1, \ldots, v_d, z_1, \ldots, z_d \rangle$ we introduce:

- the canonical metabolic[2] bilinear form $\varphi_0(v_i, z_j) = \delta_{i,j}$, $\varphi_0(v_i, v_j) = \varphi_0(z_i, z_j) = 0$ and
- the bilinear skew symmetric product given by the formula

$$[z_i, \mathfrak{n}] = 0 \quad \text{and} \quad [v_i, v_j] = \sum_{k=1}^{d} a^i_{kj} z_k$$

where the structure constants are determined by a family of d-quadratic matrices $A_i = (a^i_{kj})$ for $1 \leq i \leq d$.

Then, $(\mathfrak{n}(A_1, \ldots, A_d), \varphi_0)$ is a 2-step quadratic Lie algebra.

[2] A nondegenerate symmetric bilinear form on a vector space V is called metabolic if it has a totally isotropic subspace of dimension a half of the dimension of V. In characteristic not 2, the classes of metabolic and hyperbolic forms coincide.

Current work: examples

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ 0 & -a_1 & 0 & a_4 & a_5 \\ 0 & -a_2 & -a_4 & 0 & a_6 \\ 0 & -a_3 & -a_5 & -a_6 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & -a_1 & -a_2 & -a_3 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & a_7 & a_8 \\ a_2 & 0 & -a_7 & 0 & a_9 \\ a_3 & 0 & -a_8 & -a_9 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & a_1 & 0 & -a_4 & -a_5 \\ -a_1 & 0 & 0 & -a_7 & -a_8 \\ 0 & 0 & 0 & 0 & 0 \\ a_4 & a_7 & 0 & 0 & a_{10} \\ a_5 & a_8 & 0 & -a_{10} & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0 & a_2 & a_4 & 0 & -a_6 \\ -a_2 & 0 & a_7 & 0 & -a_9 \\ -a_4 & -a_7 & 0 & 0 & -a_{10} \\ 0 & 0 & 0 & 0 & 0 \\ a_6 & a_9 & a_{10} & 0 & 0 \end{pmatrix} \quad A_5 = \begin{pmatrix} 0 & a_3 & a_5 & a_6 & 0 \\ -a_3 & 0 & a_8 & a_9 & 0 \\ -a_5 & -a_8 & 0 & a_{10} & 0 \\ -a_6 & -a_9 & -a_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B(A_1, \ldots, A_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 \\ 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 & -a_7 & -a_8 & -a_9 \\ -a_1 & 0 & a_4 & a_5 & 0 & a_7 & a_8 & 0 & 0 & -a_{10} \\ -a_2 & -a_4 & 0 & a_6 & -a_7 & 0 & a_9 & 0 & a_{10} & 0 \\ -a_3 & -a_5 & -a_6 & 0 & -a_8 & -a_9 & 0 & -a_{10} & 0 & 0 \end{pmatrix}$$

Current work: examples

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 \\ 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 & -a_7 & -a_8 & -a_9 \\ -a_1 & 0 & a_4 & a_5 & 0 & a_7 & a_8 & 0 & 0 & -a_{10} \\ -a_2 & -a_4 & 0 & a_6 & -a_7 & 0 & a_9 & 0 & a_{10} & 0 \\ -a_3 & -a_5 & -a_6 & 0 & -a_8 & -a_9 & 0 & -a_{10} & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} v_1 \wedge v_2 &= -a_1 z_3 - a_2 z_4 - a_3 z_5, & v_2 \wedge v_4 &= -a_2 z_1 + a_7 z_3 - a_9 z_5, \\ v_1 \wedge v_3 &= a_1 z_2 - a_4 z_4 - a_5 z_5, & v_2 \wedge v_5 &= -a_3 z_1 + a_8 z_3 + a_9 z_4, \\ v_1 \wedge v_4 &= a_2 z_2 + a_4 z_3 - a_6 z_5, & v_3 \wedge v_4 &= -a_4 z_1 - a_7 z_2 - a_{10} z_5, \\ v_1 \wedge v_5 &= a_3 z_2 + a_5 z_3 + a_6 z_4, & v_3 \wedge v_5 &= -a_5 z_1 - a_8 z_2 + a_{10} z_4, \\ v_2 \wedge v_3 &= -a_1 z_1 - a_7 z_4 - a_8 z_5, & v_4 \wedge v_5 &= -a_6 z_1 - a_9 z_2 - a_{10} z_3. \end{aligned}$$

Current work: examples

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 & -a_8 & -a_9 & -a_{10} \\
0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & -a_{11} & -a_{12} & -a_{13} & -a_{14} & -a_{15} & -a_{16} \\
-a_1 & 0 & a_5 & a_6 & a_7 & 0 & a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & -a_{17} & -a_{18} & -a_{19} \\
-a_2 & -a_5 & 0 & a_8 & a_9 & -a_{11} & 0 & a_{14} & a_{15} & 0 & a_{17} & a_{18} & 0 & 0 & -a_{20} \\
-a_3 & -a_6 & -a_8 & 0 & a_{10} & -a_{12} & -a_{14} & 0 & a_{16} & -a_{17} & 0 & a_{19} & 0 & a_{20} & 0 \\
-a_4 & -a_7 & -a_9 & -a_{10} & 0 & -a_{13} & -a_{15} & -a_{16} & 0 & -a_{18} & -a_{19} & 0 & -a_{20} & 0 & 0
\end{pmatrix}
$$

$$
\begin{aligned}
v_1 \wedge v_2 &= -a_1 z_3 - a_2 z_4 - a_3 z_5 - a_4 z_6, & v_1 \wedge v_3 &= a_1 z_2 - a_5 z_4 - a_6 z_5 - a_7 z_6, \\
v_1 \wedge v_4 &= a_2 z_2 + a_5 z_3 - a_8 z_5 - a_9 z_6, & v_1 \wedge v_5 &= a_3 z_2 + a_6 z_3 + a_8 z_4 - a_{10} z_6, \\
v_1 \wedge v_6 &= a_4 z_2 + a_7 z_3 + a_9 z_4 + a_{10} z_5, & v_2 \wedge v_3 &= -a_1 z_1 - a_{11} z_4 - a_{12} z_5 - a_{13} z_6, \\
v_2 \wedge v_4 &= -a_2 z_1 + a_{11} z_3 - a_{14} z_5 - a_{15} z_6, & v_2 \wedge v_5 &= -a_3 z_1 + a_{12} z_3 + a_{14} z_4 - a_{16} z_6, \\
v_2 \wedge v_6 &= -a_4 z_1 + a_{13} z_3 + a_{15} z_4 + a_{16} z_5, & v_3 \wedge v_4 &= -a_5 z_1 - a_{11} z_2 - a_{17} z_5 - a_{18} z_6, \\
v_3 \wedge v_5 &= -a_6 z_1 - a_{12} z_2 + a_{17} z_4 - a_{19} z_6, & v_3 \wedge v_6 &= -a_7 z_1 - a_{13} z_2 + a_{18} z_4 + a_{19} z_5, \\
v_4 \wedge v_5 &= -a_8 z_1 - a_{14} z_2 - a_{17} z_3 - a_{20} z_6, & v_4 \wedge v_6 &= -a_9 z_1 - a_{15} z_2 - a_{18} z_3 + a_{20} z_5, \\
v_5 \wedge v_6 &= -a_{10} z_1 - a_{16} z_2 - a_{19} z_3 - a_{20} z_4.
\end{aligned}
$$

Universal construction: algebras

Benito, de-la-Concepción, Sesma, Roldán-López

For every $d \geq 4$, there exist quadratic 2-step Lie algebras of type d. Up to isometric isomorphisms the algebras in this class are of the form $(\mathfrak{n}(A_1, \ldots, A_{d_1}), \varphi) \perp (\mathfrak{a}, \phi)$ where $d = d_1 + d_2$ and $4 \neq d_1 \geq 3$, $\{A_1, \ldots, A_{d_1}\}$ is a d_1-quadratic family of matrices, and $(\mathfrak{a}, \phi)$ is a quadratic abelian Lie algebra of dimension $d_2 \geq 0$.

Universal construction: automorphisms

The vector space $\mathfrak{v} = \{v_1, \dots, v_d\}$ provides the free nilpotent $\mathfrak{n}_{d,2} = \mathfrak{v} \oplus \Lambda^2\mathfrak{v}$ by means of the skew product $[v_i, v_j] = v_i \wedge v_j$.

Any linear map $f\colon \mathfrak{v} \to \mathfrak{n}_{d,2} = \mathfrak{v} \oplus \Lambda^2\mathfrak{v}$ for which $f(v_1), \dots, f(v_d)$ are linearly independent, extends to the automorphism τ_f by declaring

$$\tau_f([v_i, v_j]) = [f(v_i), f(v_j)] = f(v_i) \wedge f(v_j). \qquad (1)$$

In fact, any automorphism of $\mathfrak{n}_{d,2}$ is of this form. Hence the automorphisms in the basis $\{v_i, v_i \wedge v_j\}$ are given by matrices of the form:

$$\tau_f(Q, X) = \begin{pmatrix} Q & \mathbf{0}_{d \times \frac{d(d-1)}{2}} \\ X & \hat{Q} \end{pmatrix}$$

where X is a any matrix of order $\frac{d(d-1)}{2} \times d$, Q is a regular matrix of order $d \times d$, $\hat{Q}$ is a matrix completely determined from Q by following the rule (1). In case $Q = (b_{ij})$, we get

$$\tau_f(v_i \wedge v_j) = \sum_{1 \le r < s \le n} \det \begin{pmatrix} b_{ri} & b_{rj} \\ b_{si} & b_{sj} \end{pmatrix} v_r \wedge v_s.$$

Universal construction: automorphisms

Benito, de-la-Concepción, Sesma, Roldán-López

Let $\{A_1, \ldots, A_d\}$ and $\{E_1, \ldots, E_d\}$ be two families of d-quadratic matrices and $(\mathfrak{n}(A_1, \ldots, A_d), \varphi_0)$ and $(\mathfrak{n}(E_1, \ldots, E_d), \psi_0)$ be the quadratic Lie algebras attached to them as it is described in the universal construction. Then, the Lie algebras $(\mathfrak{n}(A_1, \ldots, A_d), \varphi_0)$ and $(\mathfrak{n}(E_1, \ldots, E_d), \psi_0)$ are isometrically isomorphic if and only if there exists a regular $d \times d$ matrix Q such that

$$B(E_1, \ldots, E_d) = Q^t B(A_1, \ldots, A_d)\hat{Q}.$$

where $\hat{Q}$ is given in terms of $Q = (b_{ij})$ by means of the automorphism $\tau_f(Q, 0)$.

Current work: achievements

Achievements

- Create *d*-quadratic matrices families.
- Define Lie algebra given a *d*-quadratic family.
- All 3-quadratic families describe isomorphic algebras.

Current work: goals

Noui and Revoy proved that there is a finite number of quadratic 2-step nilpotent algebras of type less or equal than 8. We are now working on automorphisms in order to improve this result. Based on the computation of a large number of random algebras we get:

Conjecture

All 5-quadratic families describe isomorphic algebras.

Conjecture

Given two d-quadratic families, where $d \geq 6$, there is not always an automorphism between them.

Thank you!