Universal enveloping of Lie and Leibniz crossed modules

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Non-associative Algebras in Cádiz

Cádiz, February 21–24, 2018

Unión Europea– Fondo Europeo de Desarrollo Regional
Ministerio de Economía, industria y Competitividad
MTM2016-79661-P Agencia Estatal de Investigación

Universal enveloping algebra of a Lie algebra

Given a Lie algebra L, its universal enveloping algebra $\mathtt{U}(L)$ has two remarkable characteristics:

1. Isomorphic representations: the category of Lie modules over L is isomorphic to the category of modules over $\mathtt{U}(L)$;
2. Universality: the functor $\mathtt{U}\colon$ **Lie** $\to$ **Alg** is left adjoint to the liezation functor $\mathtt{Lie}\colon$ **Alg** $\to$ **Lie.**

Universal enveloping algebra of a Leibniz algebra

For Leibniz algebras, these roles are played by two different functors.

Given a Leibniz algebra $\mathfrak{p}$, we have the universal enveloping algebra of a Leibniz algebra $\mathtt{UL}(\mathfrak{p})$.

Loday-Pirashvili (1993)

The category of left $\mathtt{UL}(\mathfrak{p})$-modules is isomorphic to the category of $\mathfrak{p}$-representations.

Universal enveloping dialgebra of a Leibniz algebra

If associative algebras are replaced by dialgebras:

Let $\mathtt{Lb}\colon \mathbf{Dias} \to \mathbf{Lb}$ be the functor that assigns to every dialgebra D the Leibniz bracket given by $[x, y] = x \dashv y - y \vdash x$.

Loday (2001)

The functor $\mathtt{Lb}$ admits a left adjoint, $\mathtt{U_d}\colon \mathbf{Lb} \to \mathbf{Dias}$, which assigns to a Leibniz algebra $\mathfrak{p}$ its universal enveloping dialgebra $\mathtt{U_d}(\mathfrak{p})$.

Crossed modules

Crossed modules of groups were described for the first time by Whitehead in the late 1940s as an algebraic model for path-connected CW-spaces whose homotopy groups are trivial in dimensions greater than 2.

J.H.C. Whitehead.
Combinatorial homotopy. II.
Bull. Amer. Math. Soc. **55** (1949) 453–496.

Crossed modules are essentially the same as categorical groups, called strict 2-groups, since they are categorified groups in which the group laws hold strictly, as equations.

They can be thought as a two-dimensional generalization of the concept of group.

Objectives

For crossed modules of Lie algebras or Lie 2-algebras, to construct the universal enveloping associative 2-algebra such that:

1. Isomorphic representations: the category of Lie modules over a crossed module of Lie algebras is isomorphic to the category of modules over that universal enveloping;
2. Universality: to construct a functor that is left adjoint to the 2-liezation functor.

For crossed modules of Leibniz algebras or Leibniz 2-algebras, to construct the universal enveloping associative 2-algebra such that:

1. Isomorphic representations: the category of Leibniz modules over a crossed module of Leibniz algebras is isomorphic to the category of modules over that universal enveloping.

Action in Lie algebras

Definition

Let M and P be two Lie algebras. By an action of P on M we mean a K-bilinear map $P \times M \to M,\ (p, m) \mapsto {}^{p}m$ satisfying

- ${}^{[p,p']}m = {}^{p}({}^{p'}m) - {}^{p'}({}^{p}m)$,
- ${}^{p}[m, m'] = [{}^{p}m, m'] + [m, {}^{p}m']$.

In other words, the action of P on M is a Lie homomorphism $P \to \mathrm{Der}(M)$ to the Lie algebra of derivations of M.

Crossed modules in Lie algebras

Definition

A Lie crossed module (M, P, μ) is a Lie homomorphism $\mu \colon M \to P$ together with an action of P on M such that,

- $\mu({}^{p}m) = [p, \mu(m)]$,
- ${}^{\mu(m)}m' = [m, m']$.

C. Kassel, J.-L. Loday.
Extensions centrales d'algèbres de Lie.
Ann. Inst. Fourier (Grenoble) **32** (1982) 119–142.

Crossed modules in Lie algebras

A morphism of Lie crossed modules $(\alpha, \beta) \colon (M, P, \mu) \to (M', P', \mu')$ is a pair of Lie homomorphisms $\alpha \colon M \to M'$, $\beta \colon P \to P'$ such that

- $\mu'\alpha = \beta\mu$,

$$\begin{array}{ccc} M & \xrightarrow{\mu} & P \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ M' & \xrightarrow[\mu']{} & P' \end{array}$$

- $\alpha({}^{p}m) = {}^{\beta(p)}\alpha(m)$.

We denote by **XLie** the category of crossed modules of Lie algebras.

There are several equivalent descriptions of the category **XLie**.

The equivalence between Lie crossed modules and cat^1-Lie algebras will be used in what follows.

Definition

A cat^1-Lie algebra (L_1, L_0, s, t) consists of a Lie algebra L_1 together with a Lie subalgebra L_0 and structural homomorphisms $s, t\colon L_1 \to L_0$ satisfying

- $s\mid_{L_0} = t\mid_{L_0} = \mathrm{id}_{L_0}$,
- $[\operatorname{Ker} s, \operatorname{Ker} t] = 0$.

Equivalence between Lie crossed modules and cat^1-Lie algebras

Given a Lie crossed module (M, P, μ), the corresponding cat^1-Lie algebra is $(M \rtimes P, P, s, t)$, where $s(m, p) = p$, $t(m, p) = \mu(m) + p$.

On the other hand, for a cat^1-Lie algebra (L_1, L_0, s, t) the corresponding Lie crossed module is $t\,|_{\operatorname{Ker} s} \colon \operatorname{Ker} s \to L_0$ with the action of L_0 on $\operatorname{Ker} s$ defined by the Lie bracket in L_1.

Actor crossed modules of Lie algebras are constructed, which provide an analogue of the Lie algebra of derivations and are used to define actions of crossed modules of Lie algebras.

J. M. Casas, M. Ladra.
The actor of a crossed module in Lie algebras.
Comm. Algebra **26** (1998) 2065–2089.

Actor of Lie crossed modules

Let (M, P, μ) be a Lie crossed module.

Der(P, M) denotes the K-module of all derivations from P to M, i.e. K-linear maps $d \colon P \to M$ such that $d[p, p'] = {}^{p}d(p') - {}^{p'}d(p)$.

Der(P, M) is a Lie algebra with

$$[d_1, d_2] = d_1 \mu d_2 - d_2 \mu d_1 .$$

Der(M, P, μ) is the Lie algebra of derivations of the crossed module (M, P, μ), i.e. pairs (ϕ, ψ) with $\phi \in \mathrm{Der}(M)$, $\psi \in \mathrm{Der}(P)$ such that

- $\psi\mu = \mu\phi$,
- $\phi({}^{p}m) = {}^{p}\phi(m) + {}^{\psi(p)}m$.

Actor of Lie crossed modules

The Lie homomorphism

$$\Delta\colon \mathrm{Der}(P, M) \to \mathrm{Der}(M, P, \mu), \quad d \mapsto (d\mu, \mu d),$$

together with the Lie action of $\mathrm{Der}(M, P, \mu)$ on $\mathrm{Der}(P, M)$ given by

$$^{(\phi,\psi)}d = \phi d - d\psi,$$

is a Lie crossed module called the actor of (M, P, μ), denoted by $\mathrm{Act}(M, P, \mu)$.

Actions of Lie crossed modules

Definition

An action of a Lie crossed module (H, G, ∂) on another Lie crossed module (M, P, μ) is defined to be a morphism $(\alpha, \beta)\colon (H, G, \partial) \to \mathrm{Act}(M, P, \mu)$ of Lie crossed modules, where

$$\mathrm{Act}(M, P, \mu) = (\mathrm{Der}(P, M), \mathrm{Der}(M, P, \mu), \Delta)$$

is the actor crossed module of (M, P, μ).

Actions of Lie crossed modules

Example

There is an action of a Lie crossed module (M, P, μ) on itself given by the morphism $(\alpha_\mu, \beta_\mu)\colon (M, P, \mu) \to \mathrm{Act}(M, P, \mu)$, where

- $\alpha_\mu\colon M \to \mathrm{Der}(P, M), \ \alpha_\mu(m)(p) = -{}^{p}m,$
- $\beta_\mu\colon P \to \mathrm{Der}(M, P, \mu), \ \beta_\mu(p) = (\phi_p, \psi_p),$

with $\quad \phi_p(m) = {}^{p}m \quad$ and $\quad \psi_p(p') = [p, p'].$

Action in algebras

Definition

An algebra A acts on another algebra R if, as a K-module, R has an A-bimodule structure

$A \times R \to R, (a, r) \mapsto a \cdot r, \qquad R \times A \to R, (r, a) \mapsto r \cdot a,$ and

$$a \cdot (rr') = (a \cdot r)r', \qquad (r \cdot a)r' = r(a \cdot r'), \qquad (rr') \cdot a = r(r' \cdot a).$$

Crossed modules of algebras

Definition

A crossed module of algebras (R, A, ρ) is an algebra homomorphism $\rho\colon R \to A$, together with an action of A on R, such that the following conditions hold:

- $\rho(a \cdot r) = a\rho(r), \qquad \rho(r \cdot a) = \rho(r)a$,
- $\rho(r) \cdot r' = rr' = r \cdot \rho(r')$.

P. Dedecker, A.S.-T. Lue.
A nonabelian two-dimensional cohomology for associative algebras.
Bull. Amer. Math. Soc. **72** (1966) 1044–1050.

A morphism $(\alpha, \beta)\colon (R, A, \rho) \to (R', A', \rho')$ of crossed modules is a pair of homomorphisms $(\alpha\colon R \to R', \beta\colon A \to A')$ such that

- $\rho'\alpha = \beta\rho$,
- $\alpha(a \cdot r) = \beta(a) \cdot \alpha(r)$ and $\alpha(r \cdot a) = \alpha(r) \cdot \beta(a)$.

Denote the category of crossed modules of algebras by **XAlg.**

Definition

A cat^1-algebra (A_1, A_0, σ, τ) consists of an algebra A_1 together with a subalgebra A_0 and structural homomorphisms $\sigma, \tau \colon A_1 \to A_0$ satisfying

- $\sigma\mid_{A_0} = \tau\mid_{A_0} = \mathrm{id}_{A_0}$,
- $\operatorname{Ker}\sigma \operatorname{Ker}\tau + \operatorname{Ker}\tau \operatorname{Ker}\sigma = 0$.

Equivalence between crossed modules of algebras and cat^1-algebras

Given a crossed module of algebras (R, A, ρ), the corresponding cat^1-algebra is $(R \rtimes A, A, \sigma, \tau)$, where $\sigma(r, a) = a$, $\tau(r, a) = \rho(r) + a$.

On the other hand, for a cat^1-algebra (A_1, A_0, σ, τ) the corresponding crossed module is $\tau \mid_{\mathrm{Ker}\,\sigma} : \mathrm{Ker}\,\sigma \to A_0$ with the action of A_0 on $\mathrm{Ker}\,\sigma$ defined by the multiplication in A_1.

Liezation of crossed modules of algebras

We can associate to a crossed module of algebras (R, A, ρ) the Lie crossed module $\big(\texttt{Lie}(R), \texttt{Lie}(A), \texttt{Lie}(\rho)\big)$ with the action of $\texttt{Lie}(A)$ on $\texttt{Lie}(R)$ given by ${}^{a}r = a \cdot r - r \cdot a$.

$\big(\texttt{Lie}(R), \texttt{Lie}(A), \texttt{Lie}(\rho)\big)$ is a Lie crossed module.

Moreover, this assignment defines a functor $\texttt{XLie}\colon$ **XAlg** $\to$ **XLie** which is a natural generalization of the functor $\texttt{Lie}\colon$ **Alg** $\to$ **Lie** in the following sense.

Liezation of crossed modules of algebras

There are full embeddings

$$\mathbb{I}_0, \mathbb{I}_1 : \ \textbf{Alg} \longrightarrow \textbf{XAlg} \qquad\qquad \mathbb{I}'_0, \mathbb{I}'_1 : \ \textbf{Lie} \longrightarrow \textbf{XLie}$$

defined,

- for an algebra A by $\mathbb{I}_0(A) = (0, A, 0)$, $\mathbb{I}_1(A) = (A, A, \mathrm{id}_A)$,
- for a Lie algebra P by $\mathbb{I}'_0(P) = (0, P, 0)$, $\mathbb{I}'_1(P) = (P, P, \mathrm{id}_P)$.

It is immediate to see that we have the following commutative diagram

$$\begin{array}{ccc} \textbf{Alg} & \xrightarrow{\mathbb{I}_i} & \textbf{XAlg} \\ {\scriptstyle \mathtt{Lie}} \downarrow & & \downarrow {\scriptstyle \mathtt{XLie}} \\ \textbf{Lie} & \xrightarrow{\mathbb{I}'_i} & \textbf{XLie} \end{array}$$

for $i = 0, 1$.

Universal enveloping crossed module

Now we construct a left adjoint functor to the functor $\mathtt{XLie}$, which generalizes the universal enveloping algebra functor $\mathtt{U}\colon$ **Lie** $\to$ **Alg** to crossed modules.

Given a Lie crossed module $\mu\colon M \to P$, consider its corresponding cat^1-Lie algebra $M \rtimes P \overset{s}{\underset{t}{\rightrightarrows}} P$.

Then, applying the universal enveloping algebra functor $\mathtt{U}$ we obtain a diagram of algebras

$$\mathtt{U}(M \rtimes P) \overset{\mathtt{U}(s)}{\underset{\mathtt{U}(t)}{\rightrightarrows}} \mathtt{U}(P) .$$

It is clear that $\mathtt{U}(s)\mid_{U(P)} = \mathtt{U}(t)\mid_{U(P)} = \mathrm{id}_{\mathtt{U}(P)}$, but the kernel condition in the definition of a cat^1-algebra is not fulfilled in general.

Universal enveloping crossed module

Therefore, we consider a new diagram

$$\mathrm{U}(M \rtimes P)/X \overset{\overline{\mathrm{U}}(s)}{\underset{\overline{\mathrm{U}}(t)}{\rightrightarrows}} \mathrm{U}(P) \; ,$$

where $X = \mathrm{Ker}\, \mathrm{U}(s)\, \mathrm{Ker}\, \mathrm{U}(t) + \mathrm{Ker}\, \mathrm{U}(t)\, \mathrm{Ker}\, \mathrm{U}(s)$, $\overline{\mathrm{U}}(s)$ and $\overline{\mathrm{U}}(t)$ are induced by $\mathrm{U}(s)$ and $\mathrm{U}(t)$, respectively.

This is a cat^1-algebra.

Define $\mathrm{XU}(M, P, \mu)$ as the crossed module of algebras corresponding to the cat^1-algebra, that is,

$$\mathrm{XU}(M, P, \mu) = \left(\mathrm{Ker}\, \overline{\mathrm{U}}(s), \mathrm{U}(P), \overline{\mathrm{U}}(t)\,|_{\mathrm{Ker}\, \overline{\mathrm{U}}(s)} \right).$$

Note that, in fact, $\mathrm{XU}(M, P, \mu)$ is an object of **XAlg**.

Universal enveloping crossed module

Definition

Given a Lie crossed module (M, P, μ), the crossed module of algebras $\mathrm{XU}(M, P, \mu)$ is called the universal enveloping crossed module of (M, P, μ).

Universal enveloping crossed module

It is easy to see that the universal enveloping crossed module construction provides a functor $\mathtt{XU}\colon \mathbf{XLie} \to \mathbf{XAlg}$, which is a natural generalization of the functor $\mathtt{U}$ in the sense that the diagram

$$\begin{array}{ccc} \mathbf{Lie} & \xrightarrow{\mathbb{I}_0'} & \mathbf{XLie} \\ {\scriptstyle \mathtt{U}}\downarrow & & \downarrow{\scriptstyle \mathtt{XU}} \\ \mathbf{Alg} & \xrightarrow{\mathbb{I}_0} & \mathbf{XAlg} \end{array}$$

is commutative.

$$\begin{array}{ccc} \textbf{Lie} & \xrightarrow{\mathbb{I}_1'} & \textbf{XLie} \\ {\scriptstyle \mathrm{U}}\downarrow & & \downarrow{\scriptstyle \mathrm{XU}} \\ \textbf{Alg} & \xrightarrow{\mathbb{I}_1} & \textbf{XAlg} \end{array}$$

Proposition

There is a natural isomorphism of functors

$$\mathbb{I}_1 \circ \mathrm{U} \cong \mathrm{XU} \circ \mathbb{I}_1'.$$

Adjunction between **XLie** and **XAlg**

The following result is a natural generalization of the well-known classical adjunction between the categories **Lie** and **Alg.**

Theorem

The functor `XU` *is left adjoint to the Liezation functor* `XLie`.

We identify a module over a Lie crossed module (H, G, μ) with a split extension in **XLie**

$$(0,0,0) \longrightarrow (M,P,\mu) \longrightarrow (H',G',\partial') \rightleftarrows (H,G,\partial) \longrightarrow (0,0,0)$$

where the kernel (M, P, μ) is an abelian crossed module of Lie algebras, that is, M, P abelian Lie algebras and trivial action of P on M, ${}^{p}m = 0$, for all $p \in P$, $m \in M$.

Modules over crossed modules of algebras

Let be an abelian crossed module of algebras $\delta\colon V \to W$, that is, δ a K-homomorphism of K-modules.

Then the K-module $\mathrm{Hom}_K(W, V)$ is an algebra with the multiplication given, for all $d_1, d_2 \in \mathrm{Hom}_K(W, V)$, by

$$d_1 * d_2 = d_1 \delta d_2.$$

Let $\mathrm{End}(V, W, \delta)$ denote the algebra of all pairs (ϕ, ψ) with $\phi \in \mathrm{End}_K(V)$ and $\psi \in \mathrm{End}_K(W)$ such that $\psi\delta = \delta\phi$. The map

$$\theta\colon \mathrm{Hom}_K(W, V) \to \mathrm{End}(V, W, \delta), \quad d \mapsto (d\delta, \delta d),$$

is a homomorphism of algebras.

Modules over crossed modules of algebras

Lemma

There is an algebra action of $\text{End}(V, W, \delta)$ *on* $\text{Hom}_K(W, V)$ *given by*

$$(\phi, \psi) \cdot d = \phi d \qquad \text{and} \qquad d \cdot (\phi, \psi) = d\psi,$$

for all $d \in \text{Hom}_K(W, V)$ *and* $(\phi, \psi) \in \text{End}(V, W, \delta)$.

Moreover, together with this action $(\text{Hom}_K(W, V), \text{End}(V, W, \delta), \theta)$ *is a crossed module of algebras.*

Modules over crossed modules of algebras

Definition

Let (R, A, ρ) be a crossed module of algebras. A left (R, A, ρ)-module is an abelian crossed module of algebras (V, W, δ) together with a morphism $(R, A, \rho) \to (\mathrm{Hom}_K(W, V), \mathrm{End}(V, W, \delta), \theta)$ of crossed modules of algebras.

Isomorphism of categories of modules

Theorem

Let (M, P, μ) be a Lie crossed module. Then there is an isomorphism of categories of left (M, P, μ)-modules and left $\mathrm{XU}(M, P, \mu)$-modules.

J. M. Casas, R. Fernández-Casado, E. Khmaladze, M. Ladra.
Universal enveloping crossed module of a Lie crossed module.
Homology Homotopy Appl. **16** (2014) 143–158.

Leibniz algebras

Definition

A (right) Leibniz algebra $\mathfrak{p}$ over K is a K-module together with a bilinear operation $[\ ,\ ] \colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$, called the Leibniz bracket, which satisfies the Leibniz identity:

$$[[p_1, p_2], p_3] = [p_1, [p_2, p_3]] + [[p_1, p_3], p_2].$$

A morphism of Leibniz algebras is a K-linear map that preserves the bracket.

We denote by **Lb** the category of Leibniz algebras and morphisms of Leibniz algebras.

Leibniz algebras

A. Bloh.
A generalization of the concept of a Lie algebra.
Sov. Math. Dokl. **6** (1965) 1450–1452.

J.-L. Loday.
Une version non commutative des algèbres de Lie: les algèbres de Leibniz.
Enseign. Math. (2) **39** (1993) 269–293.

Universal enveloping algebra of the Leibniz algebra

Let $\mathfrak{p}^l$ and $\mathfrak{p}^r$ be two copies of a Leibniz algebra $\mathfrak{p}$.

We denote by x_l and x_r the elements of $\mathfrak{p}^l$ and $\mathfrak{p}^r$ corresponding to $x \in \mathfrak{p}$.

Consider the tensor K-algebra $\mathrm{T}(\mathfrak{p}^l \oplus \mathfrak{p}^r)$, which is associative and unital.

Let I be the two-sided ideal corresponding to the relations:

$$
\begin{aligned}
&[x, y]_r = x_r y_r - y_r x_r, \\
&[x, y]_l = x_l y_r - y_r x_l, \\
&(y_r + y_l) x_l = 0.
\end{aligned}
$$

Universal enveloping algebra of the Leibniz algebra

Definition

The universal enveloping algebra of the Leibniz algebra $\mathfrak{p}$ is the associative and unital algebra

$$\mathtt{UL}(\mathfrak{p}) := T(\mathfrak{p}^l \oplus \mathfrak{p}^r)/I.$$

This construction defines a functor $\mathtt{UL}\colon$ **Lb** $\to$ **Alg.**

J.-L. Loday, T. Pirashvili.
Universal enveloping algebras of Leibniz algebras and (co)homology.
Math. Ann. **296** (1993) 139–158.

Universal enveloping algebra of the Leibniz algebra

Theorem (Loday-Pirashvili, 1993)

The category of representations of the Leibniz algebra $\mathfrak{p}$ is isomorphic to the category of left modules over $\mathtt{UL}(\mathfrak{p})$.

Crossed modules of Leibniz algebras

Definition

A crossed module of Leibniz algebras (or *Leibniz crossed module*) $(\mathfrak{q}, \mathfrak{p}, \eta)$ is a morphism of Leibniz algebras $\eta\colon \mathfrak{q} \to \mathfrak{p}$ together with an action of $\mathfrak{p}$ on $\mathfrak{q}$ such that

- $\eta([p, q]) = [p, \eta(q)]$ and $\eta([q, p]) = [\eta(q), p]$,
- $[\eta(q_1), q_2] = [q_1, q_2] = [q_1, \eta(q_2)]$.

Crossed modules of Leibniz algebras

A morphism of Leibniz crossed modules (φ, ψ) from $(\mathfrak{q}, \mathfrak{p}, \eta)$ to $(\mathfrak{q}', \mathfrak{p}', \eta')$ is a pair of Leibniz homomorphisms, $\varphi \colon \mathfrak{q} \to \mathfrak{q}'$ and $\psi \colon \mathfrak{p} \to \mathfrak{p}'$, such that

- $\psi\eta = \eta'\varphi$,
- $\varphi([p, q]) = [\psi(p), \varphi(q)]$ and $\varphi([q, p]) = [\varphi(q), \psi(p)]$.

We will denote by **XLb** the category of Leibniz crossed modules.

Functor liezation

The standard functor liezation, $\mathtt{Liez}\colon \mathbf{Lb} \to \mathbf{Lie}$, $\mathfrak{p} \mapsto \mathtt{Liez}(\mathfrak{p})$, where $\mathtt{Liez}(\mathfrak{p}) = \frac{\mathfrak{p}}{\langle [p,p] \rangle}$ can be extended to crossed modules $\mathtt{XLiez}\colon \mathbf{XLb} \to \mathbf{XLie}$.

Given a Leibniz crossed module $(\mathfrak{q}, \mathfrak{p}, \eta)$ its liezation $\mathtt{XLiez}(\mathfrak{q}, \mathfrak{p}, \eta)$ is defined as the crossed module $\left(\frac{\mathtt{Liez}(\mathfrak{q})}{[\mathfrak{q}, \mathfrak{p}]_{\mathrm{x}}}, \mathtt{Liez}(\mathfrak{p}), \overline{\eta}\right)$, where $[\mathfrak{q}, \mathfrak{p}]_{\mathrm{x}}$ is the ideal generated by the elements $[q,p] + [p,q],\ p \in \mathfrak{p},\ q \in \mathfrak{q}$.

Representations of a Leibniz crossed module

For the category of crossed modules of Lie algebras, representations can be defined via an object called the actor. However this is not the case for Leibniz crossed modules. Nevertheless, it is possible to give a definition by equations:

Definition

A representation of a Leibniz crossed module $(\mathfrak{q}, \mathfrak{p}, \eta)$ is an abelian Leibniz crossed module (N, M, μ) endowed with:

(i) Actions of the Leibniz algebra $\mathfrak{p}$ (and so $\mathfrak{q}$ via η) on N and M, such that the homomorphism μ is $\mathfrak{p}$-equivariant, that is

$$\mu([p, n]) = [p, \mu(n)],$$
$$\mu([n, p]) = [\mu(n), p],$$

for all $n \in N$ and $p \in \mathfrak{p}$.

Definition (cont.)

(ii) Two K-bilinear maps $\xi_1 : \mathfrak{q} \times M \to N$ and $\xi_2 : M \times \mathfrak{q} \to N$ such that

$$\begin{aligned}
\mu\xi_2(m, q) &= [m, q], \\
\mu\xi_1(q, m) &= [q, m], \\
\xi_2(\mu(n), q) &= [n, q], \\
\xi_1(q, \mu(n)) &= [q, n], \\
\xi_2(m, [p, q]) &= \xi_2([m, p], q) - [\xi_2(m, q), p], \\
\xi_1([p, q], m) &= \xi_2([p, m], q) - [p, \xi_2(m, q)], \\
\xi_2(m, [q, p]) &= [\xi_2(m, q), p] - \xi_2([m, p], q), \\
\xi_1([q, p], m) &= [\xi_1(q, m), p] - \xi_1(q, [m, p]), \\
\xi_2(m, [q, q']) &= [\xi_2(m, q), q'] - [\xi_2(m, q'), q], \\
\xi_1([q, q'], m) &= [\xi_1(q, m), q'] - [q, \xi_2(m, q')], \\
\xi_1(q, [p, m]) &= -\xi_1(q, [m, p]), \\
[p, \xi_1(q, m)] &= -[p, \xi_2(m, q)],
\end{aligned}$$

for all $q, q' \in \mathfrak{q}$, $p \in \mathfrak{p}$, $n \in N$, $m, m' \in M$.

Universal enveloping crossed module of algebras

Let $(\mathfrak{q}, \mathfrak{p}, \eta)$ be a Leibniz crossed module and consider its corresponding cat[1]-Leibniz algebra

$$\mathfrak{q} \rtimes \mathfrak{p} \underset{t}{\overset{s}{\rightrightarrows}} \mathfrak{p} \ ,$$

with $s(q,p) = p$ and $t(q,p) = \eta(q) + p$ for all $(q,p) \in \mathfrak{q} \rtimes \mathfrak{p}$.

Now, if we apply $\mathtt{UL}$ to the previous diagram, we get

$$\mathtt{UL}(\mathfrak{q} \rtimes \mathfrak{p}) \underset{\mathtt{UL}(t)}{\overset{\mathtt{UL}(s)}{\rightrightarrows}} \mathtt{UL}(\mathfrak{p}) \ .$$

Universal enveloping crossed module of algebras

In general, the second condition for cat^1-algebras is not satisfied.

Nevertheless, we can consider the quotient $\overline{\mathtt{UL}}(\mathfrak{q} \rtimes \mathfrak{p}) = \mathtt{UL}(\mathfrak{q} \rtimes \mathfrak{p})/\mathcal{X}$, where $\mathcal{X} = \operatorname{Ker} \mathtt{UL}(s) \operatorname{Ker} \mathtt{UL}(t) + \operatorname{Ker} \mathtt{UL}(t) \operatorname{Ker} \mathtt{UL}(s)$, and the induced morphisms $\overline{\mathtt{UL}}(s)$ and $\overline{\mathtt{UL}}(t)$.

In this way, the diagram

$$\overline{\mathtt{UL}}(\mathfrak{q} \rtimes \mathfrak{p}) \underset{\overline{\mathtt{UL}}(t)}{\overset{\overline{\mathtt{UL}}(s)}{\rightrightarrows}} \mathtt{UL}(\mathfrak{p})$$

is clearly a cat^1-algebra.

Universal enveloping crossed module of algebras

Definition

Define $\mathtt{XUL}(\mathfrak{q}, \mathfrak{p}, \eta)$ as the crossed module of associative algebras given by $(\mathrm{Ker}\,\overline{\mathtt{UL}}(s), \mathtt{UL}(\mathfrak{p}), \overline{\mathtt{UL}}(t)|_{\mathrm{Ker}\,\overline{\mathtt{UL}}(s)})$.

This construction defines a functor $\mathtt{XUL}\colon\ \mathbf{XLb} \to \mathbf{XAlg}$.

There are full embeddings

$$\mathtt{I}_0, \mathtt{I}_1\colon\ \mathbf{Alg} \longrightarrow \mathbf{XAlg} \qquad\qquad \mathtt{J}_0, \mathtt{J}_1\colon\ \mathbf{Lb} \longrightarrow \mathbf{XLb}$$

defined,

- for an algebra A by $\mathtt{I}_0(A) = (\{0\}, A, 0)$, $\mathtt{I}_1(A) = (A, A, \mathrm{id}_A)$,
- for a Leibniz algebra $\mathfrak{p}$ by $\mathtt{J}_0(\mathfrak{p}) = (\{0\}, \mathfrak{p}, 0)$, $\mathtt{J}_1(\mathfrak{p}) = (\mathfrak{p}, \mathfrak{p}, \mathrm{id}_{\mathfrak{p}})$.

Universal enveloping crossed module of algebras

The functor $\mathtt{XUL}$: **XLb** $\to$ **XAlg** is a natural generalization of the functor $\mathtt{UL}$, in the sense that it makes the following diagram commute,

$$\begin{array}{ccc}
\textbf{Lb} & \xrightarrow{\mathtt{J}_0} & \textbf{XLb} \\
{\scriptstyle \mathtt{UL}}\downarrow & & \downarrow{\scriptstyle \mathtt{XUL}} \\
\textbf{Alg} & \xrightarrow[\mathtt{I}_0]{} & \textbf{XAlg}
\end{array}$$

Regarding the embeddings $\mathtt{I}_1$ and $\mathtt{J}_1$, we have the following result.

$$\begin{array}{ccc} \mathbf{Lb} & \xrightarrow{\mathtt{J}_1} & \mathbf{XLb} \\ {\scriptstyle \mathtt{UL}}\downarrow & & \downarrow{\scriptstyle \mathtt{XUL}} \\ \mathbf{Alg} & \xrightarrow[\mathtt{I}_1]{} & \mathbf{XAlg} \end{array}$$

Proposition

There is a natural isomorphism of functors

$$\mathtt{XUL} \circ \mathtt{J}_1 \cong \mathtt{I}_1 \circ \mathtt{UL}\,.$$

Isomorphism between the categories of representations

Theorem

The category of representations of a Leibniz crossed module $(\mathfrak{q}, \mathfrak{p}, \eta)$ *is isomorphic to the category of left modules over its universal enveloping crossed module of algebras* $\mathtt{XUL}(\mathfrak{q}, \mathfrak{p}, \eta)$.

Loday-Pirashvili category

Loday and Pirashvili introduced a very interesting way to see Leibniz algebras as Lie algebras over another tensor category different from K-Mod, the tensor category of linear maps, denoted by $\mathcal{LM}$.

Definition

Let M and $\mathfrak{g}$ be K-modules. The objects in $\mathcal{LM}$ are K-module homomorphisms $(M \xrightarrow{\alpha} \mathfrak{g})$.

Given two objects $M \xrightarrow{\alpha} \mathfrak{g}$ and $N \xrightarrow{\beta} \mathfrak{h}$, an arrow is a pair of K-module homomorphisms $\varrho_1 : M \to N$ and $\varrho_2 : \mathfrak{g} \to \mathfrak{h}$ such that $\beta \circ \varrho_1 = \varrho_2 \circ \alpha$.

An associative algebra in $\mathcal{LM}$ is an object $(A \xrightarrow{\beta} R)$ where R is an associative K-algebra, A is a R-bimodule and β is a homomorphism of R-bimodules.

A Lie algebra in $\mathcal{LM}$ is an object $(M \xrightarrow{\alpha} \mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra, M is a right $\mathfrak{g}$-representation and α is $\mathfrak{g}$-equivariant.

A Leibniz algebra $\mathfrak{p}$ can be viewed as a Lie algebra object in $\mathcal{LM}$, namely $\mathfrak{p} \to \mathtt{Liez}(\mathfrak{p})$.

Given a Lie algebra object $(M \xrightarrow{\alpha} \mathfrak{g})$ in $\mathcal{LM}$, its universal enveloping algebra in $\mathcal{LM}$ is $\mathtt{U}(\mathfrak{g}) \otimes M \to \mathtt{U}(\mathfrak{g})$, $1 \otimes m \mapsto \alpha(m)$.

Lie crossed module in Loday-Pirashvili category

Let $(\varrho_1, \varrho_2)\colon (N, \mathfrak{h}) \to (M, \mathfrak{g})$ be a Lie crossed module in $\mathcal{LM}$.
We construct the semidirect product in $\mathcal{LM}$ by obtaining a Lie object $(N \oplus M, \mathfrak{h} \rtimes \mathfrak{g})$, where $[(n, m), (h, g)] = ([n, h] + [n, g] + \xi(m, h), [m, g])$, and $\xi\colon M \otimes \mathfrak{h} \to N$ is a part of the action of $(M, \mathfrak{g})$ on $(N, \mathfrak{h})$.

Let (s_1, s_2) and (t_1, t_2) be two arrows in $\mathcal{LM}$

$$\begin{array}{ccc} N \oplus M & \overset{s_1}{\underset{t_1}{\rightrightarrows}} & M \\ {\scriptstyle \beta\oplus\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathfrak{h} \rtimes \mathfrak{g} & \overset{s_2}{\underset{t_2}{\rightrightarrows}} & \mathfrak{g} \end{array}$$

where

- $s_1(n, m) = m, \quad s_2(h, g) = g$ and
- $t_1(n, m) = \varrho_1(n) + m, \quad t_2(h, g) = \varrho_2(h) + g$.

We apply the universal enveloping algebra functor in $\mathcal{LM}$ to the previous diagram.

$$\begin{array}{ccc}
\mathrm{U}(\mathfrak{h} \rtimes \mathfrak{g}) \otimes (N \oplus M) & \underset{\mathrm{U}(t_1)}{\overset{\mathrm{U}(s_1)}{\rightrightarrows}} & \mathrm{U}(\mathfrak{g}) \otimes M \\
\downarrow {\scriptstyle \mathrm{U}(\beta \oplus \alpha)} & & \downarrow {\scriptstyle \mathrm{U}(\alpha)} \\
\mathrm{U}(\mathfrak{h} \rtimes \mathfrak{g}) & \underset{\mathrm{U}(t_2)}{\overset{\mathrm{U}(s_2)}{\rightrightarrows}} & \mathrm{U}(\mathfrak{g})
\end{array}$$

Universal enveloping algebra of Lie crossed module in Loday-Pirashvili category

Theorem

Let $(\varrho_1, \varrho_2)\colon (N, \mathfrak{h}) \to (M, \mathfrak{g})$ *be a Lie crossed module in* $\mathcal{LM}$*.*
The following square illustrates a crossed module of algebras in $\mathcal{LM}$ *with the induced actions*

$$\begin{array}{ccc} \operatorname{Ker} \overline{\mathrm{U}}(s_1) & \xrightarrow{\overline{\mathrm{U}}(t_1)} & \mathrm{U}(\mathfrak{g}) \otimes M \\ \Big\downarrow{\scriptstyle \overline{\mathrm{U}}(\beta \oplus \alpha)} & & \Big\downarrow{\scriptstyle \mathrm{U}(\alpha)} \\ \operatorname{Ker} \overline{\mathrm{U}}(s_2) & \xrightarrow{\overline{\mathrm{U}}(t_2)} & \mathrm{U}(\mathfrak{g}) \end{array}$$

Moreover, it is the universal enveloping crossed module of algebras of $(\varrho_1, \varrho_2)\colon (N, \mathfrak{h}) \to (M, \mathfrak{g})$ *in* $\mathcal{LM}$*.*

Universal enveloping algebra of Leibniz crossed modules

Let $(\mathfrak{q}, \mathfrak{p}, \eta)$ be a crossed module of Leibniz algebras. It can be viewed as a crossed module of Lie algebras in $\mathcal{LM}$ as a square

$$\begin{array}{ccc}
\mathfrak{q} & \xrightarrow{\eta} & \mathfrak{p} \\
\downarrow & & \downarrow \\
\dfrac{\mathtt{Liez}(\mathfrak{q})}{[\mathfrak{q}, \mathfrak{p}]_{\mathtt{x}}} & \xrightarrow{\overline{\eta}} & \mathtt{Liez}(\mathfrak{p})
\end{array}$$

We can consider its universal enveloping algebra, obtaining that way a crossed module of algebras in $\mathcal{LM}$

$$\begin{array}{ccc}
\mathrm{Ker}\, \overline{\mathtt{U}}(s_1) & \xrightarrow{\overline{\mathtt{U}}(t_1)} & \mathtt{U}\left(\mathtt{Liez}(\mathfrak{p})\right) \otimes \mathfrak{p} \\
\downarrow & & \downarrow \\
\mathrm{Ker}\, \overline{\mathtt{U}}(s_2) & \xrightarrow{\overline{\mathtt{U}}(t_2)} & \mathtt{U}\left(\mathtt{Liez}(\mathfrak{p})\right)
\end{array}$$

Universal enveloping algebra of Leibniz crossed modules

We obtain its corresponding crossed module of associative algebras in the classical setting

$$\Big(\operatorname{Ker}\overline{\mathtt{U}}(s_1)\oplus\operatorname{Ker}\overline{\mathtt{U}}(s_2),\ \big(\mathtt{U}(\mathtt{Liez}(\mathfrak{p}))\otimes\mathfrak{p}\big)\oplus\mathtt{U}(\mathtt{Liez}(\mathfrak{p})),\ \big(\overline{\mathtt{U}}(t_1),\overline{\mathtt{U}}(t_2)\big)\Big).$$

Universal enveloping algebra of Leibniz crossed modules

Theorem

Let $(\mathfrak{q}, \mathfrak{p}, \eta)$ *be a crossed module of Leibniz algebras. Its universal enveloping crossed module of algebras* $\mathtt{XUL}(\mathfrak{q}, \mathfrak{p}, \eta)$ *is isomorphic to the crossed module of algebras defined above.*

R. Fernández-Casado, X. García-Martínez, M. Ladra.
A natural extension of the universal enveloping algebra functor to crossed modules of Leibniz algebras.
Appl. Categor. Struct. **25** (2017) 1059–1076.

Thank you very much for your attention