Jordan Isomorphisms of Finitary Incidence Algebras

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Contents

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Introduction

- P a set;
- $\leq$ a binary relation on P.

Definition 1.1

The relation $\leq$ is called a quasiorder if $\leq$ is reflexive and transitive. The pair $(P, \leq)$ is called a quasiordered set.

Synonyms: preorder (resp. preordered set).

Definition 1.2

A quasiorder $\leq$ is called a partial order, if it is antisymmetric. Then $(P, \leq)$ is a partially ordered set (poset).

Locally finite quasiordered sets

- $(P, \leq)$ a quasiordered set.

Definition 1.3

$(P, \leq)$ is said to be locally finite, if for every pair $x \leq y$ the set

$$[x, y] := \{z \in P \mid x \leq z \leq y\}$$

is finite.

Example 1.4

$\mathbb{N}$ with the usual partial order is locally finite, but $\mathbb{N} \cup \{\infty\}$ is not locally finite.
$\mathbb{R}$ with the usual partial order is not locally finite.

Incidence algebras

- $(P, \leq)$ a locally finite quasiordered set;
- R a commutative associative unital ring;

Definition 1.5

The incidence algebra of P over R is the set of functions

$$I(P,R) = \{f : P \times P \to R \mid f(x,y) = 0, \text{ if } x \not\leq y\}$$

with the natural R-module structure and multiplication given by the convolution

$$(fg)(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y).$$

It is associative (in general, non-commutative) unital R-algebra.

Examples

- The full matrix algebra $M_n(R)$
  - $P = \{1, \dots, n\}$ with $x \leq y$ for all $x, y \in P$;
- The upper triangular matrix algebra $T_n(R)$
  - $P = \{1, \dots, n\}$ with the usual partial order;

Remark 1.6

If $|P| = n$, then $I(P, R)$ is isomorphic to a subalgebra of $M_n(R)$, and by this reason $I(P, R)$ is sometimes called a structural matrix algebra over R. Moreover, if $\leq$ is a partial order, then $I(P, R)$ can be identified with a subalgebra of $T_n(R)$.

Alternative description of $I(P,R)$

- $(P,\leq)$ a locally finite quasiordered set;
- R a commutative associative unital ring;

Remark 1.7

$I(P,R)$ is the set of formal series $\{\alpha = \sum_{x\leq y} \alpha_{xy} e_{xy} \mid \alpha_{xy} \in R\}$, where e_{xy} is a symbol and

$$\left(\sum_{x\leq y} \alpha_{xy} e_{xy}\right)\left(\sum_{x\leq y} \beta_{xy} e_{xy}\right) = \sum_{x\leq y}\left(\sum_{x\leq z\leq y} \alpha_{xz}\beta_{zy}\right) e_{xy}.$$

In particular, if P is finite, then $I(P,R)$ is the semigroup algebra of $\{e_{xy} \mid x\leq y\}\cup\{0\}$, where $e_{xy}e_{uv} = \delta_{yu}e_{xv}$.

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Jordan homomorphisms

- A and B algebras over a commutative ring R;
- $\varphi : A \to B$ an R-linear map.

Definition 1.8

The map φ is called a Jordan homomorphism, if φ preserves the Jordan product, i.e. $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$, where $a \circ b = ab + ba$. A bijective Jordan homomorphism is called a Jordan isomorphism.

Examples of Jordan homomorphisms

- Homomorphisms $A \to B$;
- anti-homomorphisms $A \to B$;
- sums of a homomorphism $\psi : A \to B$ and an anti-homomorphism $\theta : A \to B$, provided that $\psi(a)\theta(b) = \theta(a)\psi(b) = 0$ for all $a, b \in A$.

Jordan isomorphisms on simple algebras

Theorem 1.9 (Ancochea [2])

Each Jordan automorphism of the quaternion algebra Q is either an automorphism or an anti-automorphism.

Theorem 1.10 (Ancochea [3])

Each Jordan automorphism of a division algebra D of characteristic different from 2 is either an automorphism or an anti-automorphism.

Theorem 1.11 (Ancochea [3])

Each Jordan automorphism of a simple algebra A of characteristic different from 2 is either an isomorphism or an anti-isomorphism.

In particular, this is true for the full matrix algebra $M_n(D)$ over a division ring D.

Eliminating the restriction on the characteristic

Kaplansky and Hua considered linear maps $\varphi : A \to A'$ satisfying

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a), \quad (1)$$
$$\varphi(1_A) = \varphi(1_{A'}). \quad (2)$$

If $\mathrm{char}\, A \neq 2$, then (1) and (2) are equivalent to $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$ for all a, b.

Theorem 1.12 (Hua [7])

An additive bijective map φ from a division ring D into itself satisfying (1) and (2) is either an automorphism, or an anti-automorphism.

Theorem 1.13 (Kaplansky [9])

A linear bijective map φ between unital simple algebras A and A' satisfying (1) and (2) is either an isomorphism, or an anti-isomorphism.

If the $\mathrm{char}\, A \neq 2$, this recovers the above mentioned results of Ancochea.

Jordan homomorphisms on prime rings

- R a ring of characteristic different from 2;
- R' a prime ring of characteristic different from 2 and 3;
- $\varphi : R \to R'$ is a Jordan homomorphism.

Theorem 1.14 (Herstein [6])

If φ is onto, then φ is either a homomorphism, or an anti-homomorphism.

Theorem 1.15 (Smiley [12])

The Herstein's result holds for R' of characteristic 3 as well.

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Jordan homomorphisms $M_n(R) \to A$

By a Jordan homomorphism between two rings R and R' Jacobson and Rickart meant an additive map $\varphi : R \to R'$ which satisfies

1. $\varphi(a^2) = \varphi(a)^2$,
2. $\varphi(aba) = \varphi(a)\varphi(b)\varphi(a)$.

It follows from (1) that φ preserves the Jordan product. If R is 2-torsionfree, then the converse also holds, and (2) is also satisfied in this case.

- R an arbitrary unital ring;
- $n \geq 2$.

Theorem 1.16 (Jacobson-Rickart [8])

Each Jordan homomorphism of the rings $M_n(R) \to A$ is the sum of a homomorphism and an anti-homomorphism.

Theorem 1.17 (Molnár-Šemrl [11])

Each Jordan automorphism of $T_n(\mathbb{C})$ is either an automorphism, or an anti-automorphism.

Jordan isomorphisms of $T_n(R)$

- R a 2-torsionfree commutative unital ring;
- $n \geq 2$.

Theorem 1.18 (Beidar-Brešar-Chebotar [4])

The following conditions are equivalent:

1. *R is connected (i.e. $E(R) = \{0, 1\}$);*
2. *each Jordan isomorphism $T_n(R) \to A$ is either an isomorphism, or an anti-isomorphism.*

Near-sum

- $A = A_0 \oplus A_1$, as an R-module, where A_0 is a subalgebra of A and A_1 is an ideal of A;
- $\psi, \theta : A \to B$ are a homomorphism and $\theta : A \to B$ is an anti-homomorphism;
- $\psi|_{A_0} = \theta|_{A_0}$ and $\psi(a)\theta(b) = \theta(a)\psi(b) = 0$ for all $a, b \in A_1$.

Definition 1.19 (Benkovič [5])

The near-sum of ψ and θ (with respect to A_0 and A_1) is the R-linear map $\varphi : A \to B$, which satisfies

1. $\varphi|_{A_0} = \psi|_{A_0} = \theta|_{A_0}$;
2. $\varphi|_{A_1} = \psi|_{A_1} + \theta|_{A_1}$.

Proposition 1.20 (Benkovič [5])

The near-sum of a homomorphism and an anti-homomorphism is a Jordan homomorphism.

Jordan homomorphisms of $T_n(R)$

- R a 2-torsionfree commutative unital ring;
- $n \geq 2$;
- $D_n(R)$ the subalgebra of $T_n(R)$ consisting of the diagonal matrices;
- $S_n(R)$ the ideal of $T_n(R)$ consisting of the strictly upper triangular matrices.

Theorem 1.21 (Benkovič [5])

Each Jordan homomorphism $\varphi : T_n(R) \to A$ is the near-sum of a homomorphism $\psi : T_n(R) \to A$ and an anti-homomorphism $\theta : T_n(R) \to A$ with respect to $D_n(R)$ and $S_n(R)$.

- R a 2-torsionfree commutative unital ring;
- $n \geq 2$;
- $(P, \leq)$ either a finite poset, or a finite quasi-ordered set each of whose equivalence classes contains at least 2 elements;
- $D(P, R)$ the subalgebra of $I(P, R)$ consisting of the diagonal elements;
- $S(P, R)$ the ideal of $I(P, R)$ consisting of the elements with zero on the diagonal.

Theorem 1.22 (Akkurt-Akkurt-Barker [1])

Each Jordan homomorphism $\varphi : I(P, R) \to A$ is the near sum of a homomorphism $\psi : I(P, R) \to A$ and an anti-homomorphism $\theta : I(P, R) \to A$ with respect to $D(P, R)$ and $S(P, R)$.

Finitary incidence algebras

- $(P, \leq)$ a (not necessarily locally finite) poset;
- R a commutative associative unital ring;

Remark 1.23

$I(P, R)$ is an R-module, but not an algebra, since the convolution $\alpha\beta$ of $\alpha, \beta \in I(P, R)$ may be undefined.

Definition 1.24 (Khripchenko and Novikov [10])

An element $\alpha = \sum_{x \leq y} \alpha_{xy} e_{xy} \in I(P, R)$ is called a finitary series if for every $x \leq y$ the set

$$\{(u, v) \mid x \leq u < v \leq y,\ \alpha_{uv} \neq 0\}$$

is finite.

Proposition 1.25 (Khripchenko and Novikov [10])

The set of finitary series, denoted by $FI(P, R)$, forms an algebra under convolution. It is called the finitary incidence algebra of P over R. Moreover, $I(P, R)$ is a bimodule over $FI(P, R)$.

Remark 1.26

If P is locally finite, then $I(P,R) = FI(P,R)$.

Theorem 1.27 (Khripchenko-Novikov [10])

If R is a field, then $FI(P,R) \cong FI(Q,R) \Rightarrow P \cong Q$.

Corollary 1.28

If R is a field and P is not locally finite, then there is no locally finite Q, such that $FI(P,R) \cong I(Q,R)$.

$D(P, R)$ and $Z(P, R)$

Definition 1.29

An element $\alpha \in FI(P, R)$ is said to be *diagonal*, if $\alpha_{xy} = 0$ for $x \neq y$. Diagonal elements form a commutative subalgebra of $FI(P, R)$, which we denote by $D(P, R)$.

Definition 1.30

The elements $\alpha \in I(P, R)$ satisfying $\alpha_{xx} = 0$ for all x form an $FI(P, R)$-submodule of $I(P, R)$ denoted by $Z(P, R)$. Consequently, $FZ(P, R) := Z(P, R) \cap FI(P, R)$ is an ideal of $FI(P, R)$.

Proposition 1.31

The R-module admits the decomposition $I(P, R) = D(P, R) \oplus Z(P, R)$. Consequently, $FI(P, R) = D(P, R) \oplus FZ(P, R)$ as an R-module.

Jordan isomorphisms of $FI(P, R)$

The subalgebra $\tilde{I}(P, R)$

Definition 2.1

Denote by $\tilde{I}(P, R)$ the subalgebra of $FI(P, R)$ consisting of the finite sums $\alpha = \sum_{x \leq y} \alpha_{xy} e_{xy}$.

Definition 2.2

We introduce $\tilde{D}(P, R) = \tilde{I}(P, R) \cap D(P, R)$ and $\tilde{Z}(P, R) = \tilde{I}(P, R) \cap Z(P, R)$.

Proposition 2.3

The subset $\tilde{D}(P, R) = \tilde{I}(P, R) \cap D(P, R)$ is a subalgebra of $\tilde{I}(P, R)$ and $\tilde{Z}(P, R) = \tilde{I}(P, R) \cap Z(P, R)$ is an ideal of $\tilde{I}(P, R)$. Moreover, $\tilde{I}(P, R) = \tilde{D}(P, R) \oplus \tilde{Z}(P, R)$, as an R-module.

The restriction of a Jordan isomorphism to $\tilde{I}(P, R)$

- $(X, \leq)$, an arbitrary (non-necessarily locally finite) poset;
- R is a commutative 2-torsionfree unital ring;
- A an associative R-algebra;
- φ a Jordan homomorphism from $FI(P, R)$ to A.

Proposition 2.4

The restriction of φ to $\tilde{I}(P, R)$ is a Jordan homomorphism $\tilde{I}(P, R) \to A$. The proof of Theorem 2.1 from [1] works in this case, resulting that the R-linear maps

$$\psi(e_{xy}) = \varphi(e_x)\varphi(e_{xy})\varphi(e_y),$$
$$\theta(e_{xy}) = \varphi(e_y)\varphi(e_{xy})\varphi(e_x)$$

which are, respectively, a homomorphism and an anti-homomorphism $\tilde{I}(P, R) \to A$. Moreover, $\varphi|_{\tilde{I}(P,R)}$ is the near-sum of ψ and θ with respect to the subalgebra $\tilde{D}(P, R)$ and the ideal $\tilde{Z}(P, R)$ of $\tilde{I}(P, R)$.

Natural questions

Problem 2.5

Can ψ and θ be extended to a homomorphism and an anti-homomorphism $FI(P,R) \to A$, respectively?

Problem 2.6

Will φ be the near-sum of the extensions of ψ and θ with respect to $D(P,R)$ and $Z(P,R)$?

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Key lemmas

- $\varphi : FI(P,R) \to A$ a Jordan homomorphism.

Lemma 2.7

For any $f \in FI(P,R)$ one has

$$\forall x < y: \ \alpha_{xy}\varphi(e_{xy}) = \varphi(e_x)\varphi(\alpha)\varphi(e_y) + \varphi(e_y)\varphi(\alpha)\varphi(e_x),$$
$$\forall x: \ \alpha_{xx}\varphi(e_x) = \varphi(e_x)\varphi(\alpha)\varphi(e_x).$$

Lemma 2.8

Let φ be bijective. Then, given $a, b \in A$, one has $a = b$ is and only if

$$\begin{cases} \forall x < y: & \varphi(e_x)a\varphi(e_y) + \varphi(e_y)a\varphi(e_x) = \varphi(e_x)b\varphi(e_y) + \varphi(e_y)b\varphi(e_x), \\ \forall x: & \varphi(e_x)a\varphi(e_x) = \varphi(e_x)b\varphi(e_x). \end{cases}$$

ψ and θ on $D(P, R)$

- φ a Jordan isomorphism from $FI(P, R)$ to A.

Proposition 2.9

Let $\varphi : FI(P, R) \to A$ be a Jordan isomorphism. Then $\varphi|_{D(P,R)}$ is a homomorphism (and an anti-homomorphism at the same time).

An extension of ψ

- φ a Jordan isomorphism from $FI(P,R)$ to A.

Lemma 2.10

Given $\alpha \in FZ(P,R)$ and $x \le y$, define

$$\alpha'_{xy} = \varphi^{-1}(\varphi(e_x)\varphi(\alpha)\varphi(e_y))_{xy}.$$

Then $\alpha' \in FZ(P,R)$.

Definition 2.11

Given $\alpha \in FZ(P,R)$ and $x \le y$, set $\tilde{\psi}(\alpha) = \varphi(\alpha')$. In the general situation, when $\alpha \in FI(P,R)$, write $\alpha = \alpha_D + \alpha_Z$ and thus set $\tilde{\psi}(\alpha) = \varphi(\alpha_D) + \tilde{\psi}(\alpha_Z)$.

Lemma 2.12

The map $\tilde{\psi}$ is an R-linear extension of ψ.

$\tilde{\psi}$ is a homomorphism

Lemma 2.13

If $\alpha \in D(P, R)$ and $\beta \in FZ(P, R)$, then $\tilde{\psi}(\alpha\beta) = \tilde{\psi}(\alpha)\tilde{\psi}(\beta)$. Similarly, if $\alpha \in FZ(P, R)$ and $\beta \in D(P, R)$, then $\tilde{\psi}(\alpha\beta) = \tilde{\psi}(\alpha)\tilde{\psi}(\beta)$.

Lemma 2.14

If $\alpha, \beta \in FZ(P, R)$, then $\tilde{\psi}(\alpha\beta) = \tilde{\psi}(\alpha)\tilde{\psi}(\beta)$.

Proposition 2.15

The map $\tilde{\psi}$ is a homomorphism $FI(P, R) \to A$.

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An extension of θ

- φ a Jordan isomorphism from $FI(P,R)$ to A.

Proposition 2.16

Given $\alpha \in FZ(P,R)$ and $x \le y$, define

$$\alpha''_{xy} = \varphi^{-1}(\varphi(e_y)\varphi(\alpha)\varphi(e_x))_{xy}.$$

Then $\alpha'' \in FZ(P,R)$.

Definition 2.17

Given $\alpha \in FZ(P,R)$ and $x \le y$, set $\tilde{\psi}(\alpha) = \varphi(\alpha'')$. In the general situation, when $\alpha \in FI(P,R)$, write $\alpha = \alpha_D + \alpha_Z$ and thus set $\tilde{\theta}(\alpha) = \varphi(\alpha_D) + \tilde{\theta}(\alpha_Z)$.

Lemma 2.18

The map $\tilde{\theta}$ is an anti-homomorphism $FI(P,R) \to A$ which extends θ.

The decomposition of φ

Theorem 2.19

Each Jordan isomorphism $\varphi : FI(P,R) \to A$ is the near-sum of $\tilde{\psi}$ and $\tilde{\theta}$ with respect to the subalgebra $D(P,R)$ and the ideal $FZ(P,R)$.

Future work

Dropping the assumptions on P and R

Problem 3.1

Prove the above mentioned result without the restriction that R is 2-torsionfree (i.e. do not use the result of Akkurt *et al* [1]).

Problem 3.2

Generalize the description of Jordan isomorphism to the case, when P is quasi-ordered.

Generalizing the result by Beidar-Brešar-Chebotar

- P a poset;
- R connected.

Problem 3.3

Is it true that each Jordan isomorphism $FI(P, R) \to A$ is either an isomorphism or an anti-isomorphism?

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