On ternary algebras and a (new) ternary generalization of Jordan algebras

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Outline

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Definition (Kurosh, 1969)

Let $\mathcal{A}$ be a vector space over a field $\mathbb{F}$. $\mathcal{A}$ is said to be an Ω-algebra over $\mathbb{F}$ if Ω is a system of multilinear algebraic operations defined on $\mathcal{A}$,

$$\Omega = \{\omega_i : |\omega_i| = n_i \in \mathbb{N}, i \in I\},$$

where $|\omega_i|$ denotes the arity of ω_i.

Remark

Hereinafter, we write

$$[x_1, \ldots, x_{n_i}]_i$$

instead of

$$\omega_i(x_1, \ldots, x_{n_i}).$$

Definition

An Ω-algebra $\mathcal{A}$ is said to be anticommutative if all n_i-ary operations ($n_i \geq 2$), $[., \ldots, .]_i \in \Omega$ are anticommutative, that is, if

$$[x_1, \ldots, x_{n_i}]_i = sgn(\sigma) \left[x_{\sigma(1)}, \ldots, x_{\sigma(n_i)}\right]_i$$

for all $x_1, \ldots, x_{n_i} \in \mathcal{A}$, $\sigma \in \mathcal{S}_n$ (the symmetric permutation group).

Remark

If $char(\mathbb{F}) \neq 2$, $[., \ldots, .]_i \in \Omega$ is anticommutative if and only if it is null whenever a pair of its entries are equal.

Examples

- *n*-ary Filippov algebras (formerly known as *n*-Lie algebras);
- *n*-ary Mal'tsev algebras (2nd part of the talk).

Definition

An Ω*-algebra* $\mathcal{A}$ *is said to be* *totally commutative* *if all* n_i*-ary operations* ($n_i \geq 2$), $[.,\ldots,.]_i \in \Omega$ *are totally commutative, that is, if*

$$[x_1,\ldots,x_{n_i}]_i = \left[x_{\sigma(1)},\ldots,x_{\sigma(n_i)}\right]_i,$$

for all $x_1,\ldots,x_{n_i} \in \mathcal{A}$, $\sigma \in \mathcal{S}_n$.

Examples

- *n*-ary Jordan algebras are totally commutative (3rd part of the talk);
- Jordan triple systems are only partially commutative.

Definition

An Ω-algebra $\mathcal{A}$ is said to be totally associative *if all n_i-ary operations ($n_i \geq 2$), $[., \ldots, .]_i \in \Omega$ are totally associative, that is, if*

$$[x_1, \ldots, [x_j, \ldots, x_{j+n_i-1}], \ldots x_{2n_i-1}]_i = [x_1, \ldots, [x_k, \ldots, x_{k+n_i-1}], \ldots x_{2n_i-1}]_i,$$

for all $j, k \in \{1, \ldots, n_i\}$, $x_1, \ldots, x_{2n_i-1} \in \mathcal{A}$.

Ω-algebras with more than one operation

Example

A Sabinin algebra $\mathcal{A}$ is a vector space with two infinite systems of multilinear operators:

$$\Omega_1 = \{\langle x_1, \ldots, x_n; y, z\rangle\} \text{ the } \textit{Mikheev-Sabinin brackets}$$

abstract versions of the covariant derivatives $\nabla_{x_1} \ldots \nabla_{x_n} T(x, y)$ of the torsion tensor of the canonical connection of a loop satisfying certain identities (see Mostovoy, Perez-Izquierdo and Shestakov, 2014);

$$\Omega_2 = \{[x_1, \ldots, x_n; y_1, \ldots, y_m]\} \; n \geq 1, \; m \geq 2$$

Ω-algebras with more than one operation

Example

A Poisson algebra $(\mathcal{A}, \{.,.\}, \cdot)$ is a vector space $\mathcal{A}$ with two multilinear operators such that:

1. $(\mathcal{A}, \{.,.\})$ is a Lie algebra;
2. $(\mathcal{A}, \cdot)$ is commutative and associative;
3. $\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b$. (the Leibniz identity)

If, in addition,

$$\{x, y\} \cdot \{z, u\} + \{z, x\} \cdot \{y, u\} + \{y, z\} \cdot \{x, u\} = 0,$$

is satisfied, $(\mathcal{A}, \{.,.\}, \cdot)$ is said to be a Poisson-Farkas algebra (Farkas, 1998).

Let $\mathcal{A}$ be an Ω-algebra over $\mathbb{F}$. For every multilinear n_i-ary multiplication

$$[., \ldots, .]_i \in \Omega,$$

for each $j = 1, ..., n$, and for all $x_1, \ldots, \hat{x}_j, \ldots, x_{n_i} \in \mathcal{A}$, define the multiplication operator

$$M_j(x_1, \ldots, \hat{x}_j, \ldots, x_{n_i})$$

as the linear mapping $\quad y \mapsto [x_1, \ldots, y, \ldots, x_{n_i}]_i \,.$

Example

Given an Ω-algebra $\mathcal{A}$ with one n-ary multiplication, $[., \ldots, .]$, for each $x = (x_2, \ldots, x_n)$, $x_i \in \mathcal{A}$, the operator $R_x = R_{(x_2,\ldots,x_n)}$ defined by

$$y \mapsto R_x(y) = [y, x_2, \ldots, x_n]$$

is called a right multiplication operator.

Definition

Let $\mathcal{A}$ be an Ω-algebra over $\mathbb{F}$.

1. *The* multiplication algebra *of $\mathcal{A}$ is the subalgebra $M = M(\mathcal{A})$ of the linear transformation algebra $End_{\mathbb{F}}(\mathcal{A})$ generated by all multiplications and by $Id_{\mathcal{A}}$.*
2. *A subspace $\mathcal{I}$ of $\mathcal{A}$ that is invariant under the $M(\mathcal{A})$ is called an* ideal *of $\mathcal{A}$.*
3. *The* square *of $\mathcal{A}$ is the following subalgebra $\mathcal{A}^2 = \langle M\ \mathcal{A}\rangle_{\mathbb{F}}$.*
4. *An Ω-algebra $\mathcal{A}$ is said to be* simple *if $\mathcal{A}^2 \neq 0$ and $\mathcal{A}$ contains no ideals distinct from 0 and $\mathcal{A}$.*
5. *The* centroid *of $\mathcal{A}$ is the following subalgebra of $End_{\mathbb{F}}(\mathcal{A})$:*

$$\Gamma(\mathcal{A}) = \left\{\phi \in End_{\mathbb{F}}(\mathcal{A}) : \phi\left([a_1, ..., a_{n_i}]_i\right) = \left[a_1, ..., \phi(a_j), ..., a_{n_i}\right]_i, \text{ for all } a_1, ..., a_{n_i} \in \mathcal{A}, j \in N_{n_i}\right\}$$

$\Gamma(\mathcal{A})$ is an associative algebra with unity over $\mathbb{F}$.

Theorem (Pozhidaev, 1999)

If $\mathcal{A}$ is a simple Ω-algebra, then $\Gamma(\mathcal{A})$ is a field.

Definition

An Ω-algebra $\mathcal{A}$ is said to be central if $\Gamma(\mathcal{A}) = \mathbb{F}$.

Remark

Hereinafter, all Ω-algebras considered will have only one multilinear n-ary operation. From now on, we will say

"an n-ary algebra"

instead of

"an Ω-algebra with one n-ary operation".

Definition

Given an arbitrary class of n-ary algebras, $\mathcal{A} = (\mathbb{V}, [., \ldots, .])$, $n > 2$, let us fix $a \in \mathbb{V}$, and for each $i \in \{1, \ldots, n\}$, define an $(n-1)$-ary algebra denoted by $\mathcal{A}_{i,a}$, by putting

$$[x_2, \ldots, x_n]_{i,a} = [x_2, \ldots, \underbrace{a}_{i\text{-th entry}}, \ldots, x_n], \; x_2, \ldots, x_n \in \mathbb{V}.$$

Each algebra $\mathcal{A}_{i,a}$, defined on $\mathbb{V}$, is called a reduced algebra of $\mathcal{A}$.

Remark

If $[., \ldots, .]$ is anticommutative or totally commutative, it is enough to write $\mathcal{A}_a$ and

$$[x_2, \ldots, x_n]_a = [a, x_2, \ldots, x_n], \; x_2, \ldots, x_n \in \mathbb{V}.$$

Example

- reduced algebras of n-ary totally (anti)commutative algebras are $(n-1)$-ary totally (anti)commutative algebras;
- reduced algebras of n-ary totally associative algebras are $(n-1)$-ary totally associative algebras.

Definition

Let $\mathcal{A} = (\mathbb{V}, [., \ldots, .])$ be an n-ary algebra. A linear map $D : \mathbb{V} \longrightarrow \mathbb{V}$ is said to be a derivation of $\mathcal{A}$ if it satisfies

$$D([x_1, \ldots, x_n]) = \sum_{i=1}^{n} [x_1, \ldots, D(x_i), \ldots, x_n], \; x_1, ..., x_n \in \mathbb{V}.$$

Definition (Kaygorodov, 2014)

Let $\mathcal{A} = (\mathbb{V}, [., \ldots, .])$ be an n-ary algebra. An $(n+1)$-ary derivation of $\mathcal{A}$ is a collection

$$(f_0, \ldots, f_n) \in End_{\mathbb{F}}(\mathbb{V})^{n+1}$$

such that $$f_0([x_1, \ldots, x_n]) = \sum_{i=1}^{n} [x_1, \ldots, f_i(x_i), \ldots, x_n], \; x_1, ..., x_n \in \mathbb{V}.$$

Remark

If $\psi_1, \ldots, \psi_n \in \Gamma(\mathcal{A})$ and $D \in Der(\mathcal{A})$, then

$$\left(\sum \psi_i, \psi_1, \ldots, \psi_n\right) \text{ and } (D, \ldots, D)$$

are $(n+1)$-derivations of $\mathcal{A}$. These and their linear combinations are said to be trivial.

Definition

Let $\mathcal{A}$ be an n-ary anticommutative algebra over a field $\mathbb{F}$, with multiplication $[., \ldots, .]$. Let $\mathcal{R}$ and $Lie(\mathcal{R})$ be, resp., the vector space and the Lie algebra generated by the right multiplication operators by elements of $\mathcal{A}$. A quasi-derivation of $\mathcal{A}$ is every operator $D : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$[D, R_x] \in Lie(\mathcal{R}), \text{ for all } R_x \in Lie(\mathcal{R})$$

The set of all quasi-derivations of $\mathcal{A}$ is denoted by $QDer(\mathcal{A})$.

We have: $Der(\mathcal{A}) \subseteq QDer(\mathcal{A}) \subseteq End_{\mathbb{F}}(\mathcal{A})$.

Outline

1. Introduction
   - n-ary algebras
   - Filippov algebras
2. n-ary Mal'tsev algebras
   - Building the notion of n-ary Mal'tsev algebra
   - First results
   - Ternary Mal'tsev algebras arising on composition algebras
3. n-ary Jordan algebras
   - Jordan algebras and its generalizations. The notion of n-ary Jordan algebra
   - Ternary algebras with a generalized multiplication
   - The simple ternary Jordan algebra $\mathbb{A}$
   - Other examples

Definition (Filippov, 1985)

Let $\mathcal{L}$ be an algebra over $\mathbb{F}$ equipped with a mutilinear n-ary $(n \geq 2)$ multiplication $[., \ldots, .]$. $\mathcal{L}$ is said to be an n-ary Filippov algebra (also known as n-Lie algebra) if it is anticommutative and if, for every $x_1, \ldots, x_n, y_2, \ldots, y_n \in \mathcal{L}$,

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n]. \quad (1)$$

Remark

1. *(1) is called generalized Jacobi identity (GJI, for short);*
2. *An n-ary Filippov algebra is said to be perfect if it coincides with its square;*
3. *An n-ary algebra whose multilinear multiplication satisfies the GJI is called an n-ary Leibniz algebra.*

Example (Filippov, 1985)

Let L be a real Euclidean $(n+1)$-dimensional $(n \geq 2)$ vector space equipped with $[., \dots, .]$, denoting the vector cross product of n elements in L. Fix an o.n. basis $\mathcal{E}_{n+1} = \{e_1, \dots, e_{n+1}\}$ of L, we have:

$$[x_1, \dots, x_n] = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} & e_1 \\ x_{21} & x_{22} & \cdots & x_{2n} & e_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(n+1)1} & x_{(n+1)2} & \cdots & x_{(n+1)n} & e_{n+1} \end{vmatrix}$$

where for $i \in \{1, \dots, n\}$, $x_i = (x_{1i}, x_{2i}, \dots, x_{(n+1)i})$ on the fixed basis.

Example (cont.)

By n-linearity the following multiplication table is enough to define $[\cdot, \cdots, \cdot]$:

$$[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = (-1)^{n+1+i} e_i, \quad i \in \{1, \ldots, n+1\}, \tag{2}$$

$\hat{e}_i$: e_i omitted. The other products are null or obtained from (2) by anticommutativity. $L := A_{n+1}$ is the n-ary Filippov algebra of vector cross product (a natural generalization of the binary vector cross product in $\mathbb{R}^3$).

Classification of anticommutative n-ary algebras of dim. $\leq (n+1)$

Admit that $char(\mathbb{F}) = 0$.

- Up to isomorphism, there is only one n-dimensional n-ary anticommutative algebra:

  $$A_n : \qquad [e_1, ..., e_n] = e_1;$$

  A_n is a Filippov algebra and any n-dimensional n-ary anticommutative algebra is isomorphic to A_n.

- Up to isomorphism, there is only one perfect $(n+1)$-dimensional n-ary Filippov algebra:

  $$A_{n+1} : \qquad [e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = (-1)^{n+1+i} e_i;$$

- Let L be an n-ary $(n+1)$-dimensional algebra with basis $\{e_1, ..., e_{n+1}\}$. Let

$$e^i = (-1)^{n+1+i}[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}], \ i = 1, \ldots, n+1;$$

Then, the multiplication in L is defined by a square matrix B such that:

$$\left(e^1, \ldots, e^{n+1}\right) = \left(e_1, \ldots, e_{n+1}\right) B.$$

In this case, L is an n-ary Filippov algebra iff B is symmetric.

Theorem (Filippov, 1985)

The reduced algebras of n-ary Filippov algebras are $(n-1)$-ary Filippov algebras.

Theorem

The reduced algebras of n-ary Leibniz algebras are $(n-1)$-ary Leibniz algebras.

Filippov algebras and Nambu Mechanics

- The theory of n-ary Filippov algebras has a close relation with Nambu Mechanics (Nambu, 1973): a proposal of generalization to obtain generalized Hamiltonian equations of the movement.
- Specifically, Filipov algebras are the implicit algebraic concept underlying this theory.
- In Nambu Mechanics the Nambu parenthesis $\{\cdot,\cdot,\cdot\}$ is defined in the Euclidian phase space $\mathbb{R}^3$ by the determinant

$$\{f_1, f_2, f_3\} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \left| \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} \right|$$

- This multiplication is trilinear and anticommutative and generalizes the Poisson parenthesis to the ternary case.
- It also satisfies a so-called fundamental identity:

$$\{\{g,h,f_1\},f_2,f_3\}+\{f_1,\{g,h,f_2\},f_3\}+\{f_1,f_2,\{g,h,f_3\}\}=\{g,h,\{f_1,f_2,f_3\}\}$$

which is a ternary version of the GJI.

Building ternary Filippov algebras

In recent works (Pozhidaev, 2015 and 2017):

- simple ternary Filippov algebras have been built by means of Poisson algebras and Poisson-Farkas algebras;
- ternary Filippov superalgebras have been built by means of Poisson superalgebras and Poisson-Farkas superalgebras.

Outline

Definition

Let $\mathbb{V}$ be a finite dimensional vector space over a field $\mathbb{F}$, equipped with a non-singular bilinear symmetric form $\langle .,. \rangle$. We say that $\mathbb{V}$ is an n-ary algebra with vector cross product $(n \geq 2)$ if it can be defined an n-ary multilinear operation $[., \dots, .] : \mathbb{V} \times \dots \times \mathbb{V} \longrightarrow \mathbb{V}$ satisfying

1. $\langle [x_1, \dots, x_n], x_i \rangle = 0$, *for all* $x_i \in \mathbb{V}$, $i = 1, \dots, n$;
2. $\langle [x_1, \dots, x_n], [x_1, \dots, x_n] \rangle = \det\left(\langle x_i, x_j \rangle\right)$, *for all* $x_i \in \mathbb{V}$, $i = 1, \dots, n$.

Theorem (Classification Th. of n-ary algebras with v.c.p., Brown and Gray, 1967)

The only possible n-ary vector cross product algebras are:

1. *Case* $n = 2$*:*
   - *the simple 3-dimensional Lie algebra* $sl(2)$
   - *the simple 7-dimensional Mal'tsev algebra* C_7*;*
2. *Case* $n \geq 3$*:*
   - *the simple* $(n+1)$*-dimensional Filippov (n-ary) algebras* A_{n+1} *(n-ary analogues of* $sl(2)$*);*
   - *an exceptional 8-dimensional ternary algebra,* M_8*, arising on composition algebras.*

Question

Is it possible to generalize the notion of Mal'tsev algebra in such a way that it includes M_8 in the ternary case?

Definition

An algebra $\mathcal{A}$ over a field $\mathbb{F}$ is called a Mal'tsev algebra if it satisfies

1. $x^2 = 0$, *for all* $x \in \mathcal{A}$;
2. $J(x, y, xz) = J(x, y, z)x$, *for all* $x, y, z \in \mathcal{A}$, *(Mal'tsev identity) where* $J(x, y, z) = (xy)z + (yz)x + (zx)y$ *is the Jacobian.*

Rewriting the Mal'tsev identity, we have:

$$(zx)(xy) + (z(xy))x = ((zx)x)y - ((zy)x)x, \tag{3}$$

which is equivalent in $Ass(\mathcal{R})$ to

$$R_x R_{xy} + R_{xy} R_x = R_x^2 R_y - R_y R_x^2, \tag{4}$$

where $Ass(\mathcal{R})$ is the associative algebra generated by the right multiplication operators $R_x : a \mapsto ax$.

Let's look at a generalization process of the Jacobi identity. This can be written as

$$R_y R_z + R_z R_x = R_{[x,y]} = R_{xR_y}$$

and thus, in the ternary case, we have

$$R_{y,z} R_{u,v} + R_{u,v} R_{x,y} = R_{yR_{u,v},z} + R_{y,zR_{u,v}}$$

where $xR_{y,z} = R_{y,z}(x) = [x,y,z]$.
Therefore, (4) can be analogously generalized to an algebra $\mathcal{A}$ with an n-ary multiplication:

$$R_x \sum_{i=1}^{n} R_{x_2,\ldots,x_i R_y,\ldots,x_n} + \sum_{i=1}^{n} R_{x_2,\ldots,x_i R_y,\ldots,x_n} R_x = R_x^2 R_y - R_y R_x^2$$

Definition (Pozhidaev, 2001)

An n-ary algebra $\mathcal{A} = (\mathbb{V}, [., \ldots, .])$ over a field $\mathbb{F}$ is said to be an n-ary Mal'tsev algebra if its multilinear multiplication is anticommutative and it satisfies

$$\sum_{i=1}^{n} [[z, x_2, \ldots, x_n], x_2, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n] +$$

$$\sum_{i=1}^{n} [[z, x_2, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n], x_2, \ldots, x_n]$$

$$= [[[z, x_2, \ldots, \ldots, x_n], x_2, \ldots, x_n], y_2, \ldots, y_n] - [[[z, y_2, \ldots, \ldots, y_n], x_2, \ldots, x_n], x_2, \ldots, x_n].$$

This identity is known as the Generalized Mal'tsev identity (GMI).

Outline

The GMI is equivalent to

$$-J(zR_x, x_2, \ldots, x_n; y_2, \ldots, y_n) = J(z, x_2, \ldots, x_n; y_2, \ldots, y_n)R_x$$

where $J(x_1, x_2, \ldots, x_n; y_2, \ldots, y_n) = LHS(GJI) - RHS(GJI)$.

Lemma (Pozhidaev, 2001)

Every n-ary Filippov algebra is an n-ary Mal'tsev algebra.

Generalizing the fact that every Lie algebra is a Mal'tsev algebra.

Lemma (Pozhidaev, 2001)

Every reduced algebra of an n-ary Mal'tsev algebra is an $(n-1)$-ary Mal'tsev algebra.

Outline

Let $\mathbb{A}$ be a composition algebra over a field $\mathbb{F}$, $char(\mathbb{F}) \neq 2$, with

- an involution $x \mapsto \overline{x}$ and unity 1;
- a non-singular, symmetric bilinear form $\langle x, y \rangle = \frac{1}{2}(x\overline{y} + y\overline{x})$;
- a norm defined by the rule $n(a) = \langle a, a \rangle$, for each $a \in \mathbb{A}$.

Theorem (Pozhidaev, 2001)

Let $\mathbb{A}$ be a composition algebra under the above circumstances. Let $M(\mathbb{A})$ be the ternary algebra defined on $\mathbb{A}$ by the multiplication

$$[x, y, z] = (x\overline{y})z - \langle y, z \rangle x + \langle x, z \rangle y - \langle x, y \rangle z. \tag{5}$$

Then $M(\mathbb{A})$ is a ternary Mal'tsev algebra.

Theorem

Let $\mathbb{A}$ *be a composition algebra in the mentioned conditions. If* $\dim \mathbb{A} \geq 4$*, then* $M(\mathbb{A})$ *is a central simple ternary Mal'tsev algebra. Moreover,* $M(\mathbb{A})$ *will be a ternary Filippov algebra only if* $\dim \mathbb{A} = 4$ *or* $char(\mathbb{F}) = 3$*.*

Remark

Therefore, if $\dim \mathbb{A} = 8$ *and* $char(\mathbb{F}) \neq 2, 3$*, we obtain a ternary Mal'tsev algebra,* $M_8 = M(\mathbb{A})$*, which is not a 3-Filippov algebra.*

Corollary

If $\mathbb{A}$ *is a composition algebra in the above conditions, with no zero divisors, over* $\mathbb{F}$ *such that* $char(\mathbb{F}) = 3$*, then* $M(\mathbb{A})$ *is a new central simple 3-Filippov algebra.*

Theorem (Saraiva, 2003)

Let $\mathcal{M} = (\mathbb{V}, [.,.,.])$ *be a ternary Mal'tsev algebra with* $\dim(\mathbb{V}) \leq 4$ *over a field of arbitrary characteristic. Then* $\mathcal{M}$ *is a ternary Filippov algebra.*

Theorem (Saraiva, 2003)

The reduced algebras of the 8-dimensional ternary Mal'tsev algebras $M(\mathbb{A})$ *which arise by fixing the elements of an o.n. basis of* $\mathbb{A}$ *are 7-dimensional simple Mal'tsev algebras.*

Remark

Not all reduced algebras of the $M(\mathbb{A})$ are simple. Take the canonical basis of $\mathbb{A}$:

$$\mathcal{C} = \{1, a, b, ab, c, ac, bc, abc\}$$

and consider $\alpha \in \mathbb{A}$ such that $\alpha^2 = -1$. Putting $u = 1 + \alpha a$, then $M(\mathbb{A})_u = (M(\mathbb{A}), [.,.]_u)$ has nontrivial ideals.

Theorem (Pozhidaev, 2001)

Let $\mathcal{A}$ be an arbitrary n-ary vector cross product algebra. Then $\mathcal{A}$ is a central simple n-ary Mal'tsev algebra.

On the derivations of M_8

Consider the ternary ternary Mal'tsev algebra $M_8 = M(\mathbb{A})$, with the canonical basis of $\mathbb{A}$:

$$\mathcal{C} = \{1, a, b, ab, c, ac, bc, abc\} \qquad (6)$$

and the ternary multiplication given by (5).
Consider:

- $\mathcal{R}$: vector space spanned by the right multiplications of M_8;
- $Ass(\mathcal{R})$: the associative algebra generated by $\mathcal{R}$;
- $Lie(\mathcal{R})$: the Lie algebra generated by $\mathcal{R}$;
- $Der(M_8)$: the derivation algebra of M_8;
- $Innder(M_8)$: the innerderivation algebra of M_8 (i.e., $Innder(M_8) = \{D \in Der(M_8) : D \in Lie(\mathcal{R})\}$.

Proposition (Pozhidaev and Saraiva, 2006)

1. $Ass(\mathcal{R}) = M_{8,8}(\mathbb{F}) = \langle \mathcal{R}^2 \rangle$
2. $Lie(\mathcal{R}) \cong D_4$ *and* $Lie(\mathcal{R}) = \mathcal{R}$ *as vector spaces;*
3. $Der(M_8) \cong B_3$.

Recall that:

$$D_4 = \mathfrak{o}(8) = \{X \in \mathfrak{gl}(8) : XD + DX^T = 0\},\ D = \begin{bmatrix} 0 & I_4 \\ I_4 & 0 \end{bmatrix}, \dim(D_4) = 28$$

$$B_3 = \mathfrak{o}(7) = \{X \in \mathfrak{gl}(7) : XB + BX^T = 0\}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_3 \\ 0 & I_3 & 0 \end{bmatrix}, \dim(B_3) = 21.$$

Explicit basis for $Der(M_8)$ can be exhibited.

Theorem (Pozhidaev and Saraiva, 2006)

All derivations of M_8 *are inner.*

Theorem (Pozhidaev and Saraiva, 2006)

$$Der\,(M_8) = \langle [R_{x,y}, R_{x,z}] + R_{x,[y,x,z]} : \; x, y, z \in \mathbb{A} \rangle_{\mathbb{F}}.$$

Theorem (Kaygorodov, 2014)

The simple ternary Mal'tsev algebra M_8 has no nontrivial 4-ary derivations.

Remark

Indeed, all 4-ary derivations of can be written as a sum of

$$\left(\sum_{i=1}^{3} \alpha_i id, \alpha_1 id, \alpha_2 id, \alpha_3 id \right) \text{ and } (D^*, D^*, D^*, D^*)$$

where $\alpha_i \in \mathbb{F}$ and $D^* \in Der\,(M_8)$.

On the quasi-derivations of M_8

Theorem (Pozhidaev and Saraiva, 2006)

Consider the ternary Mal'tsev algebra M_8 and put $L = Lie(\mathcal{R})$. Then $\mathcal{Q}Der\,(M_8)$, the set of quasi-derivations of M_8 is such that

$$\mathcal{Q}Der\,(M_8) = \langle Id \rangle_{\mathbb{F}} \oplus L.$$

We have: $Der(M_8) \subset \mathcal{Q}Der(M_8) \subset End_{\mathbb{F}}(M_8)$.

Outline

Definition (Jordan, 1933)

A Jordan algebra is a commutative algebra $\mathcal{A}$ over a field $\mathbb{F}$, $char(\mathbb{F}) \neq 2$, such that

$$(xy)\,x^2 = x\,(yx^2)\,. \qquad \text{(Jordan identity)} \qquad (7)$$

- Among the binary generalizations of Jordan algebras we have the so-called noncommutative Jordan algebras: algebras which satisfy (7) and the flexible identity

$$(xy)x = x(yx)$$

- Noncommutative Jordan algebras include alternative algebras, Jordan algebras, quasiassociative algebras, quadratic flexible algebras and anticommutative algebras.

- *n*-ary generalizations of Jordan algebras, are restricted to the ternary case, mostly due to the works of Bremner, Peresi and Hentzel (2000, 2001 and 2007).
- Among these, is the notion of Jordan triple system:

A Jordan triple system (see Bemner and Peresi, 2007; Gnedbaye and Wambst, 2007) is a ternary algebra $\mathbb{A}$ with multiplication $[\![.,.,.]\!]$ satisfying a partial commutativity property

$$[\![x, y, z]\!] = [\![z, y, x]\!]$$

and the following identity:

$$[\![[\![x, y, z]\!], u, v]\!] + [\![z, u, [\![x, y, v]\!]]\!] = [\![x, y, [\![z, u, v]\!]]\!] + [\![z, [\![y, x, u]\!], v]\!]. \tag{8}$$

A different approach

A Lie triple algebra (Osborn, 1969, and Sidorov, 1981) is a commutative, nonassociative algebra $\mathcal{A}$ over a field $\mathbb{F}$ ($char\,(\mathbb{F}) \neq 2$) satisfying

$$\left(a, b^2, c\right) = 2b\left(a, b, c\right), \tag{9}$$

where $(a, b, c) = (ab)c - a(bc)$ is the associator.
(9) is equivalent to:

$$R_{(x,y,z)} = [R_y, [R_x, R_z]], \tag{10}$$

where $[a, b] = ab - ba$ is the commutator and R_x is a right multiplication operator.

- Every Jordan algebra is a Lie triple algebra.

- Simple Lie triple algebras containing an idempotent are Jordan algebras (Osborn, 1965).

- On a commutative algebra, $\mathcal{A}$, (10) is equivalent to

$$[R_x, R_y] \in Der(\mathcal{A}). \tag{11}$$

- Putting $D_{x,y} := [R_x, R_y]$, this means that

$$D_{x,y}(ab) = D_{x,y}(a)b + aD_{x,y}(b). \tag{12}$$

- In a Jordan algebra $\mathcal{J}$,

$$Inder\left(\mathcal{J}\right) = \left\{\sum \left[R_{x_i}, R_{y_i}\right] : x_i, y_i \in \mathcal{J}\right\},$$

- Thus, in every Jordan algebra $\mathcal{A}$, $D_{x,y}$ is a derivation of $\mathcal{A}$ (Faulkner, 1967).

Definition (n-ary Jordan algebra (Saraiva, Pozhidaev and Kaygorodov, 2017))

Let $\mathcal{A}$ be an n-ary algebra with a multilinear multiplication $[\![.,\ldots,.]\!] : \times^n \mathbb{V} \to \mathbb{V}$, where $\mathbb{V}$ is the underlying vector space. $\mathcal{A}$ is said to be an *n-ary Jordan algebra* if the multiplication is totally commutative and if

$$\left[R_{(x_2,\ldots,x_n)}, R_{(y_2,\ldots,y_n)}\right] \in Der\left(\mathcal{A}\right), \tag{13}$$

where $[.,.]$ is the commutator and $R_{(x_2,\ldots,x_n)}, R_{(y_2,\ldots,y_n)}$ are right multiplication operators:

$$y \mapsto yR_{(x_2,\ldots,x_n)} = [\![y, x_2, \ldots, x_n]\!].$$

Simplifying notations, (13) assumes the form

$$D_{x,y} = [R_x, R_y] \in Der(\mathcal{A}), \quad (D_{x,y}\text{-identity}) \tag{14}$$

meaning that

$$D_{x,y} [\![z_1, \dots, z_n]\!] = \sum_{i=1}^{n} [\![z_1, \dots, D_{x,y}(z_i), \dots, z_n]\!]. \tag{15}$$

Non-totally commutative n-ary Jordan algebras are also called $D_{x,y}$-derivation algebras.

Outline

Consider:

- $\mathbb{V}$: an n-dimensional vector space over a field $\mathbb{F}$;
- f bilinear, symmetric and nondegenerate form;
- $\mathcal{B} = \{b_1, \ldots, b_n\}$: a basis of $\mathbb{V}$ such that:

$$f(b_i, b_j) = \delta_{ij}.$$

- Define on $\mathbb{F} \oplus \mathbb{V}$ the multiplication $*$ by

$$(\alpha + u)*(\beta + v) = \alpha\beta + f(u, v) + \alpha v + \beta u, \quad \alpha, \beta \in \mathbb{F}, u, v \in \mathbb{V}.$$

Then we obtain a Jordan algebra of a symmetric bilinear form f, denoted by $J(\mathbb{V}, f)$, which is simple if $\dim \mathbb{V} > 1$ and f is nondegenerate.

Question

Is it possible to generalize this example to the ternary case?

The ternary algebra $\mathcal{V}_{f,g,h}$

Given:

- $\mathbb{V}$: an n-dimensional vector space over a field $\mathbb{F}$;
- f and h: bilinear, symmetric and nondegenerate forms;
- g: trilinear, symmetric and nondegenerate form;
- $\mathcal{B} = \{b_1, \ldots, b_n\}$: a basis of $\mathbb{V}$ such that:

$$f(b_i, b_j) = \delta_{ij}, \quad h(b_i, b_j) = \delta_{ij} \quad \text{and} \quad g(b_i, b_j, b_k) = \delta_{ijk},$$

- Define the ternary algebra $\mathcal{V}_{f,g,h} := (\mathbb{F} \oplus \mathbb{V}, [\![.,.,.]\!])$ such that:

$$[\![\alpha_1 + v_1, \alpha_2 + v_2, \alpha_3 + v_3]\!] = \tag{16}$$

$$= (\alpha_1\alpha_2\alpha_3 + \alpha_1 f(v_2, v_3) + \alpha_2 f(v_1, v_3) + \alpha_3 f(v_1, v_2) + g(v_1, v_2, v_3)) +$$
$$+ (\alpha_2\alpha_3 + h(v_2, v_3)) v_1 + (\alpha_1\alpha_3 + h(v_1, v_3)) v_2 + (\alpha_1\alpha_2 + h(v_1, v_2)) v_3.$$

The first examples

Theorem (Saraiva, Pozhidaev and Kaygorodov, 2017)

The ternary algebra $\mathcal{V}_{f,g,h}$ is a ternary Jordan algebra if f, g and h are identically zero. In the opposite case, $\mathcal{V}_{f,g,h}$ is not a ternary Jordan algebra **with the following exceptions***:*

- $\mathcal{V}_{0,0,h}$, *if* $\operatorname{char}(\mathbb{F}) = 3$ *and* $\dim \mathbb{V} = 1$;
- $\mathcal{V}_{0,g,0}$, *if* $\operatorname{char}(\mathbb{F}) = 2$ *and* $\dim \mathbb{V} = 1$;
- $\mathcal{V}_{f,0,h}$, *if* $\operatorname{char}(\mathbb{F}) = 2$;
- $\mathcal{V}_{f,g,h}$, *if* $\operatorname{char}(\mathbb{F}) = 2$ *and* $\dim \mathbb{V} = 1$.

Lemma

$\mathcal{V}_{0,0,0}$ is not simple and every subspace of $\mathbb{V}$ is an ideal of $\mathcal{V}_{0,0,0}$. Further, if $\mathcal{I}$ is a proper ideal of $\mathcal{V}_{0,0,0}$, then $\mathcal{I}$ is a subspace of $\mathbb{V}$. Among the modular ternary Jordan algebras obtained in the previous theorem, only the following are simple:

- $\mathcal{V}_{0,g,0}$, *with* $\mathrm{char}\,(\mathbb{F}) = 2$ *and* $\dim \mathbb{V} = 1$;
- $\mathcal{V}_{f,0,h}$, *with* $\mathrm{char}\,(\mathbb{F}) = 2$ *and* $\dim \mathbb{V} > 1$.

Lemma

Let D be an arbitrary derivation of $\mathcal{V}_{0,0,0}$, then

1. *if char* $(\mathbb{F}) \neq 2$, *then* $Der(\mathcal{V}_{0,0,0}) \cong End(\mathbb{V})^{(-)}$;
2. *if char* $(\mathbb{F}) = 2$ *and* $\dim \mathbb{V} = 1$, *then* $Der(\mathcal{V}_{0,0,0}) \cong End(\mathcal{V}_{0,0,0})^{(-)}$;
3. *if char* $(\mathbb{F}) = 2$ *and* $\dim \mathbb{V} > 1$, *then* $D(\mathbb{V}) \subseteq \mathbb{V}$, $Der|_{\mathbb{V}}(\mathcal{V}_{0,0,0}) \cong End(\mathbb{V})^{(-)}$ *and* $D(1)$ *may be an arbitrary element of* $\mathcal{V}_{0,0,0}$, *where* $Der|_{\mathbb{V}}(\mathcal{V}_{0,0,0})$ *is the algebra of derivations of* $\mathcal{V}_{0,0,0}$ *restricted to* $\mathbb{V}$.

Outline

- Replace h by $(.,.)$;
- Restrict $[\![\cdot,\cdot,\cdot]\!]$ to $\mathbb{V}$ and admit that *char* $(\mathbb{F})=0$;

$$[\![x,y,z]\!]=(y,z)\,x+(x,z)\,y+(x,y)\,z. \tag{17}$$

- $\mathbb{A}:=(\mathbb{V},[\![\cdot,\cdot,\cdot]\!])$

Theorem

- *$\mathbb{A}$ is a ternary Jordan algebra.*
- *The ternary Jordan algebra $\mathbb{A}$ is simple, except if* dim $\mathbb{V}=2$ *and char* $(\mathbb{F})=2$.

Remark

$\mathbb{A}$ is not a Jordan triple system.

Reduced algebras of *n*-ary Jordan algebras

Theorem

The reduced algebras of the ternary Jordan algebra $\mathbb{A}$ are not Jordan algebras.

Remark

The reduced algebras of Jordan triple systems may not be Jordan algebras.

Remark

The main subclass of n-ary Jordan algebras consists of totally commutative and totally associative n-ary algebras, in which the "reduced property" happens.

Identites of degree k in $\mathbb{A}$

An identity satisfied by a ternary algebra is said to be of degree (or level) k, with $k \in \mathbb{N}$, if k is the number of times that the multiplication appears in each term of the identity.

Lemma

All degree 1 identities on $\mathbb{A}$ are a consequence of the total commutativity of (17).

Lemma

All degree 2 identities on $\mathbb{A}$ are a consequence of the total commutativity of (17), by means of a lifting process.

Remark

Lifting: every process which allows to obtain $(k+1)$-degree identities starting form k-degree identities. This include techniques of two types:
(i) embedding – which justifies that

$$[\![[\![a,b,c]\!],d,e]\!] = [\![[\![b,a,c]\!],d,e]\!] \text{ starting from } [\![a,b,c]\!] = [\![b,a,c]\!],$$

(ii) replacing an element by a triple – justifying that

$$[\![[\![a,b,c]\!],d,e]\!] = [\![[\![a,b,c]\!],e,d]\!] \text{ starting from } [\![a,b,c]\!] = [\![a,c,b]\!].$$

Theorem

Der $(\mathbb{A}) =$ *Inder* $(\mathbb{A}) = so(n)$.

Remark

- *If a finite dimensional Lie algebra, over a field of characteristic zero has an invertible derivation, then it is a nilpotent algebra (Jacobson, 1955);*
- *If a finite dimensional Jordan algebra, over a field of characteristic zero has an invertible derivation, then it is a nilpotent algebra (Kaygorodov and Popov, 2016);*
- *this result doesn't hold for ternary Jordan algebras (a consequence of last theorem): just take the ternary Jordan algebra* $\mathbb{A}$ *with* $\dim \mathbb{V} = 4$ *and consider the derivation* $\sum_{1 \leq i < j \leq 4} (e_{ij} - e_{ji})$*. There is a simple ternary Jordan algebra with an invertible derivation.*

Outline

Ternary symmetrized matrix algebras

- Consider the following ternary algebras

$$\mathfrak{A} = \left(M_n\left(\mathbb{F}\right), [\![.,.,.]\!]\right),$$

where $[\![.,.,.]\!]$ is the anticommutator:

$$[\![A,B,C]\!] = sym\left(ABC\right) = ABC+ACB+BAC+BCA+CAB+CBA.$$

- $\mathfrak{A}$ is not a ternary Jordan algebra since

$$D_{x,y}\,[\![A,B,C]\!] = [\![D_{x,y}(A),B,C]\!]+[\![A,D_{x,y}(B),C]\!]+[\![A,B,D_{x,y}(C)]\!]$$

doesn't hold when $char\left(\mathbb{F}\right) \neq 3,\ n = 3.$ Just take:

$$x = \left(e_{23}, e_{32}\right),\ \ y = \left(e_{22}, e_{23}\right),\ \ A = e_{12},\ \ B = e_{23},\ \ C = e_{32}.$$

Ternary symmetrized matrix algebras

Theorem

Given different $i, j \in \{1, \ldots, n\}$ *the following 2-dimensional subalgebras of* $M_n(\mathbb{F})$

$$\mathfrak{S}_1 = \langle e_{ii}, e_{ij} \rangle_{\mathbb{F}} \text{ and } \qquad \mathfrak{S}_2 = \langle e_{ij}, e_{ji} \rangle_{\mathbb{F}}, \quad (i \neq j),$$

are non-isomorphic ternary Jordan subalgebras of $\mathfrak{A}$. *Further,* $\mathfrak{S}_2$ *is simple.*

Ternary algebras defined on the Cayley-Dickson algebras

Recall the Cayley-Dickson doubling process (Schafer, 1954).
Consider:

- $\mathcal{A}$: a unital algebra over a field $\mathbb{F}$, $char\,(\mathbb{F}) \neq 2$;
- an involution $x \mapsto \overline{x}$: $x + \overline{x}$, $x\overline{x} \in \mathbb{F}$, for all $x \in \mathcal{A}$;
- Let $a \in \mathbb{F} \setminus \{0\}$ and define a new algebra $(\mathcal{A}, a)$ as follows:

the underlying vector space,	$\mathcal{A} \oplus \mathcal{A}$
the addition,	$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$
the scalar multiplication,	$c\,(x_1, x_2) = (cx_1, cx_2)$
the multiplication	$(x_1, x_2)\,(y_1, y_2) = (x_1 y_1 + a y_2 \overline{x_2}, \overline{x_1} y_2 + y_1 x_2)$.

The corresponding involution is given by:

$$\overline{(x_1, x_2)} = (\overline{x_1}, -x_2)\,.$$

Ternary algebras defined on the Cayley-Dickson algebras

Starting with $\mathbb{F}$ such that *char* $(\mathbb{F}) \neq 2$, we obtain a sequence of 2^t-dimensional algebras denoted by $\mathcal{U}_t$, among which:

$\mathcal{U}_0 = \mathbb{F}$,	the scalars,
$\mathcal{U}_1 = \mathbb{C}(a) = (\mathbb{F}, a)$,	generalized complex numbers,
$\mathcal{U}_2 = \mathbb{H}(a, b) = (\mathbb{C}(a), b)$,	the generalized quaternions,
$\mathcal{U}_3 = \mathbb{O}(a, b, c) = (\mathbb{H}(a, b), c)$,	the generalized octonions

Define on each $\mathcal{U}_t$, $t = 2, 3, \ldots$ the ternary multiplication:

$$[\![x, y, z]\!] = (x\overline{y})\, z \tag{18}$$

and take $\quad \mathcal{D}_t = (\mathcal{U}_t, [\![., ., .]\!])$.

$\mathcal{D}_t$ are not ternary Jordan algebras, since $[\![\cdot, \cdot, \cdot]\!]$ is not totally commutative.

Ternary algebras defined on the Cayley-Dickson algebras

Theorem

$\mathcal{D}_2$ is a simple ternary $D_{x,y}$-derivation algebra.

Theorem

$$\textit{Consider} \qquad \mathbb{H}(a,b) = \langle 1\rangle_{\mathbb{F}} \oplus \mathbb{H}(a,b)_s.$$

Then $D \in Der(\mathcal{D}_2)$ iff there exists $\Phi, \Psi \in End\,(\mathbb{H}(a,b))$ such that

$$D = \Phi + \Psi,$$

where $\Phi \in Der(\mathbb{H}(a,b))$ and $\Psi(x) = x\Psi(1)$, for all $x \in \mathbb{H}(a,b)$ and $\Psi(1) \in \mathbb{H}(a,b)_s$.

Lemma

$\mathcal{D}_3$ is not a ternary $D_{x,y}$-derivation algebra.

An analog of the TKK-construction for ternary algebras

- Purpose: take the Tits-Kantor-Koecher (TKK) unified construction of the exceptional simple classical Lie algebras by means of a composition algebra and a degree 3 simple Jordan algebra and use an analogue construction to define TJA.
- Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a 3-graded ternary algebra with the product $[x, y, z]$. By definition, we have:

$$[L_i, L_j, L_k] \subseteq L_{i+j+k}, \text{ (modular addition in } \{-1, 0, 1\})$$

- Define a ternary operation on $\mathcal{J} := L_0$ by the rule:

$$[\![x, y, z]\!] = \mathcal{S}_{x,y,z}[[[u_{-1}, x, u_1], y, v_{-1}], z, v_1], \qquad (19)$$

where $\mathcal{S}_{x,y,z}$ is the symmetrization operator in x, y, z and $u_i, v_i \in L_i$, $i = -1, 1$.

- Consider $L = A_1$ be the simple 4-dimensional ternary Filippov algebra over $\mathbb{C}$ with the standard basis $\{e_1, e_2, e_3, e_4\}$ and the multiplication table

$$[e_1, \ldots, \hat{e}_i, \ldots, e_4] = (-1)^i e_i.$$

Change this basis in A_1 to $\{a, b, a_{-1}, a_1\}$, such that

$$a = \frac{\mathbf{i}}{2} e_1,\ b = \frac{1}{2} e_2,\ a_{-1} = e_3 - \mathbf{i} e_4,\ a_1 = e_3 + \mathbf{i} e_4,\ \text{where } \mathbf{i}^2 = -1.$$

- Then

$$A_1 = \langle a_{-1} \rangle \oplus \langle a, b \rangle \oplus \langle a_1 \rangle$$

is a 3-grading on A_1, with $\mathcal{J} = L_0 = \langle a, b \rangle$.
- Putting $u_{-1} = v_{-1} = a_{-1},\ u_1 = v_1 = a_1$ in (19), we obtain the following multiplication table in $\mathcal{J}$:

$$[\![a, a, a]\!] = 6b,\ [\![a, a, b]\!] = 2a,\ [\![a, b, b]\!] = -2b \text{ and } [\![b, b, b]\!] = -6a. \qquad (20)$$

- Then:

$$D_{(a,a),(a,b)} \doteq -3e_{12} + e_{21}, D_{(a,a),(b,b)} \doteq e_{11} - e_{22} \text{ and } D_{(a,b),(b,b)} \doteq -e_{12} + 3e_{21}$$

$\doteq$: an equality up to a scalar; e_{ij} is the matrix unit in the basis $\{a, b\}$.

- Consider a ternary totally commutative algebra $\mathcal{J}$ over an arbitrary field with the multiplication table (20). Then

$$D_{x,y} \in \langle -3e_{12} + e_{21}, e_{11} - e_{22}, -e_{12} + 3e_{21} \rangle$$

- The $D_{x,y}$-identity holds if and only if $char\,(\mathbb{F}) = 2$.

- It is also possible to define a TJA structure on L_{-1}. Put:

$$[\![x,y,z]\!] = \mathcal{S}_{x,y,z}[[[u_0,x,u_1],y,v_1],z,v_0],$$

$u_i, v_i \in L_i,\ \ i = 0,1,\ \ x,y,z \in L_{-1}$. In this case we have

$$[\![a_{-1},a_{-1},a_{-1}]\!] = a_{-1},$$

with

$$a_{-1} = e_3 - \mathbf{i}e_4,\ \ u_0 = \frac{\mathbf{i}}{4}e_1,\ \ v_0 = e_2,\ \ u_1 = v_1 = a_1 = e_3 + \mathbf{i}e_4.$$

- Every one-dimensional ternary algebra $\mathcal{J}$ is a ternary Jordan algebra, and $\mathcal{J}$ is simple if and only if $[\![\mathcal{J},\mathcal{J},\mathcal{J}]\!] \neq 0$.

¡ Gracias por vuestra atención!