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Baric and dibaric property transitions of a class of two-dimensional chains of evolution algebras

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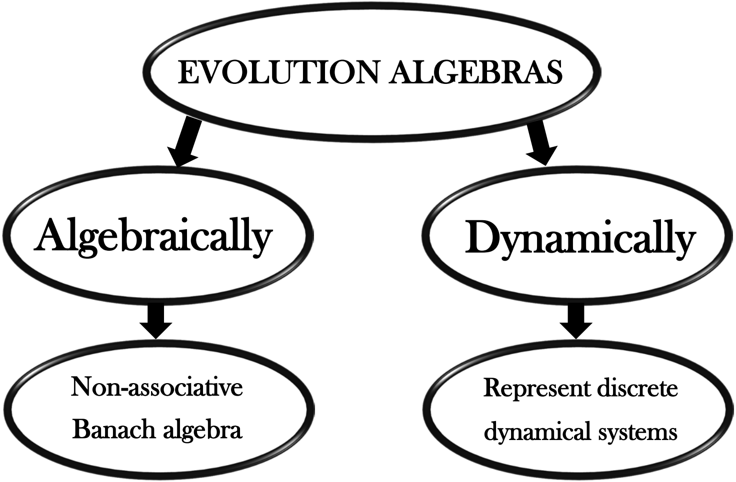
Evolution algebras

There exist several classes of non-associative algebras (baric, evolution, stochastic, etc.), whose investigation has provided a number of significant contributions to theoretical population genetics. Such classes have been defined different times by several authors, and all algebras belonging to these classes are generally called "genetic".

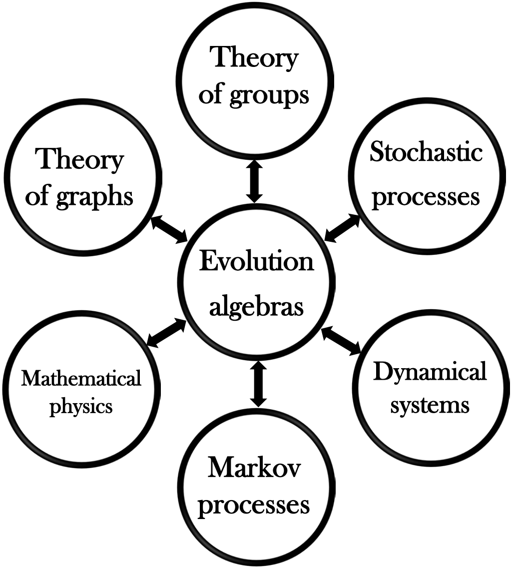
Notion of evolution algebra was introduced by J.P.Tian in [1] and the foundation of evolution algebra theory and applications in non-Mendelian genetics and Markov chains are developed.

These are algebras in which the multiplication tables are motivated by evolution laws of genetics.

Evolution algebras



Connections with other fields of science



Evolution algebras

In [1] a notion of evolution algebra is introduced and evolution algebra is defined as follows.

Definition

Let $(E, \cdot)$ be an algebra over a field K. If it admits a basis $\{e_1, e_2, \dots\}$, such that

$$e_i \cdot e_j = 0, \quad \text{if} \quad i \neq j$$

and

$$e_i \cdot e_i = \sum_k a_{ik} e_k, \quad \text{for any} \quad i,$$

then this algebra is called an *evolution algebra*. The basis is called a natural basis.

Chain of evolution algebra

Recently in [2] a notion of chain of evolution algebras is introduced. This chain is a dynamical system the state of which at each given time is an evolution algebra. The chain is defined by the sequence of matrices of the structural constants (of the evolution algebras considered in [1]) which satisfies the Chapman-Kolmogorov equation.

In this note we give the behavior of baric and dibaric properties depending on time of a class of chains of two-dimensional evolution algebras constructed in [4].

Following [2] we consider a family $\left\{E^{[s,t]}: \ s,t \in R, \ 0 \leqslant s \leqslant t\right\}$ of n-dimensional evolution algebras over the field R, with basis $e_1, \dots, e_n$ and multiplication table

$$e_i e_i = \sum_{j=1}^{n} a_{ij}^{[s,t]} e_j, \quad i = 1, \dots, n; \quad e_i e_j = 0, \quad i \neq j. \qquad (1)$$

Here parameters s, t are considered as time.

Denote by $M^{[s,t]} = \left(a_{ij}^{[s,t]}\right)_{i,j=1,\dots,n}$-the matrix of structural constants.

Chain of evolution algebra

Definition

A family $\{E^{[s,t]}: \ s,t \in R, \ 0 \leqslant s \leqslant t\}$ of n-dimensional evolution algebras over the field R is called a chain of evolution algebras (CEA) if the matrix $M^{[s,t]}$ of structural constants satisfies the Chapman-Kolmogorov equation

$$M^{[s,t]} = M^{[s,\tau]}M^{[\tau,t]}, \quad \text{for any} \quad s < \tau < t. \tag{2}$$

To construct a chain of two-dimensional evolution algebra one has to solve equation (2) for 2×2 matrix $M^{[s,t]}$. This equation gives the following system of functional equations:

$$\begin{aligned} a_{11}^{[s,t]} &= a_{11}^{[s,\tau]}a_{11}^{[\tau,t]} + a_{12}^{[s,\tau]}a_{21}^{[\tau,t]}, \\ a_{12}^{[s,t]} &= a_{11}^{[s,\tau]}a_{12}^{[\tau,t]} + a_{12}^{[s,\tau]}a_{22}^{[\tau,t]}, \\ a_{21}^{[s,t]} &= a_{21}^{[s,\tau]}a_{11}^{[\tau,t]} + a_{22}^{[s,\tau]}a_{21}^{[\tau,t]}, \\ a_{22}^{[s,t]} &= a_{21}^{[s,\tau]}a_{12}^{[\tau,t]} + a_{22}^{[s,\tau]}a_{22}^{[\tau,t]}. \end{aligned} \tag{3}$$

Examples of CEA

Here we give examples of matrices of structural constants of chains of two dimensional evolution algebras:

$$M_1^{[s,t]} = \frac{1}{2} \begin{cases} \begin{pmatrix} 1 + \frac{\Psi(t)}{\Psi(s)} & 1 - \frac{\Psi(t)}{\Psi(s)} \\ 1 - \frac{\Psi(t)}{\Psi(s)} & 1 + \frac{\Psi(t)}{\Psi(s)} \end{pmatrix}, & \text{if} \quad s \leqslant t < a; \\ \begin{pmatrix} \frac{\Psi(t)}{\Psi(s)} & -\frac{\Psi(t)}{\Psi(s)} \\ -\frac{\Psi(t)}{\Psi(s)} & \frac{\Psi(t)}{\Psi(s)} \end{pmatrix}, & \text{if} \quad t \geqslant a, \end{cases}$$

where $a > 0$ and Ψ is an arbitrary function, $\Psi(s) \neq 0$;

$$M_2^{[s,t]} = \frac{1}{2} \begin{pmatrix} \frac{h(t)+g(t)}{h(s)} & \frac{h(t)-g(t)}{h(s)} \\ \frac{h(t)+g(t)}{h(s)} & \frac{h(t)-g(t)}{h(s)} \end{pmatrix}$$

where h, g are arbitrary functions, $h(s) \neq 0$;

Examples of CEA

$$M_3^{[s,t]} = \frac{1}{2}\begin{pmatrix} h(t)\left(\frac{1}{h(s)} + g(s)\right) & h(t)\left(\frac{1}{h(s)} + g(s)\right) \\ h(t)\left(\frac{1}{h(s)} - g(s)\right) & h(t)\left(\frac{1}{h(s)} - g(s)\right) \end{pmatrix},$$

where h, g are arbitrary functions, $h(s) \neq 0$;

$$M_4^{[s,t]} = \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & 0 \\ \frac{\Phi(t)}{\psi(s)}(g(t) - g(s)) & \frac{\psi(t)}{\psi(s)} \end{pmatrix},$$

where Φ, g, ψ are arbitrary functions, $\Phi(s) \neq 0, \psi(s) \neq 0$;

$$M_5^{[s,t]} = \begin{cases} \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & 0 \\ \Phi(t)(v(t) - v(s)) & 1 \end{pmatrix}, & \text{if} \ \ s \leqslant t < b; \\ \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & 0 \\ \Phi(t)w(s) & 0 \end{pmatrix}, & \text{if} \ \ t \geqslant b, \end{cases}$$

Baric property transition

A *character* for an algebra A is a nonzero multiplicative linear form on A, that is, a nonzero algebra homomorphism from A to R [5]. Not every algebra admits a character. For example, an algebra with the zero multiplication has no character.

Definition

A pair (A, σ) consisting of an algebra A and a character σ on A is called a *baric algebra*. The homomorphism σ is called the weight (or baric) function of A and $\sigma(x)$ the weight (baric value) of x.

Baric property transition

But the evolution algebra E introduced in [1] is not baric, in general. The following theorem gives a criterion for an evolution algebra E to be baric [2].

Theorem

An n-dimensional evolution algebra E, over the field R, is baric if and only if there is a column $(a_{1i_0}, \dots, a_{ni_0})^T$ of its structural constants matrix $M = (a_{ij})_{i,j=1,\dots,n}$, such that $a_{i_0 i_0} \neq 0$ and $a_{ii_0} = 0$, for all $i \neq i_0$. Moreover, the corresponding weight function is $\sigma(x) = a_{i_0 i_0} x_{i_0}$.

Baric property transition

In [4] a notion of property transition for CEAs is defined. We recall the definitions:

Definition

Assume a CEA, $E^{[s,t]}$, has a property, say P, at pair of times (s_0, t_0); one says that the CEA has P property transition if there is a pair $(s, t) \neq (s_0, t_0)$ at which the CEA has no the property P.

Baric property transition

Denote

$$\mathcal{T} = \{(s,t) : 0 \leqslant s \leqslant t\};$$

$$\mathcal{T}_P = \{(s,t) \in \mathcal{T} : E^{[s,t]} \text{ has property } P\};$$

$$\mathcal{T}_P^0 = \mathcal{T} \setminus \mathcal{T}_P = \{(s,t) \in \mathcal{T} : E^{[s,t]} \text{ has no property } P\}.$$

The sets have the following meaning:
$\mathcal{T}_P$-the duration of the property P;
$\mathcal{T}_P^0$-the lost duration of the property P;
The partition $\{\mathcal{T}_P, \mathcal{T}_P^0\}$ of the set $\mathcal{T}$ is called P property diagram.
For example, if P =commutativity then since any evolution algebra is commutative, we conclude that any CEA has not commutativity property transition.

Baric property transition

Denote by $\mathcal{T}_b^{(i)}$ the baric property duration of the CEA $E_i^{[s,t]}$, $i = 1, \ldots, 5$.

Theorem

(i) (There is no non-baric property transition) *The CEA* $E_2^{[s,t]}$ *is not baric for any time* $(s,t) \in \mathcal{T}$;

(ii) (There is no baric property transition) *The CEA* $E_4^{[s,t]}$ *is baric for any time* $(s,t) \in \mathcal{T}$;

(iii) (There is baric property transition) *The CEAs* $E_i^{[s,t]}$, $i = 1, 3, 5$ *have baric property transition with baric property duration sets as the following*

$$\mathcal{T}_b^{(1)} = \{(s,t) \in \mathcal{T} : s \leqslant t < a,\ \Psi(s) = \Psi(t)\};$$

$$\mathcal{T}_b^{(3)} = \left\{(s,t) \in \mathcal{T} : g(s) = \pm \frac{1}{h(s)}\right\};$$

$$\mathcal{T}_b^{(5)} = \{(s,t) \in \mathcal{T} : s \leqslant t < b\} \cup \{(s,t) \in \mathcal{T} : t \geqslant b,\ \psi(s) = 0\};$$

Baric property transition

Note $\mathcal{T}_b^{(i)}$, $i = 1, 3, 5$ depend on some parameter functions and can be controlled by choosing the corresponding parameter functions $\Phi, \Psi, g, h, \psi, w$. These functions are called *baric property controllers* of the CEAs. Because, they really control the baric duration set, for example, if some of them is a strong monotone function then the duration is "minimal", i.e. the line $s = t$, but if a function is a constant function then the baric duration set is "maximal", i.e. it is $\mathcal{T}$. Since these functions are arbitrary functions, we have a rich class of controller functions, therefore we have a "powerful" control on the property to be baric.

Dibaricity of CEAs

Following [6,7]:

Definition

Let $U = \langle \omega, m \rangle_R$, denote a two-dimensional commutative algebra over R with multiplicative table

$$\omega^2 = m^2 = 0, \quad \omega \cdot m = \frac{1}{2}(\omega + m)$$

Then U is called the sex differentiation algebra.

Definition

An algebra A is called dibaric if it admits a homomorphism onto the sex differentiation algebra U.

Dibaricity of CEAs

In [3] proved the next theorem.

Let the two-dimensional evolution algebra E be given by the matrix of structural constants $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Theorem

The two dimensional real evolution algebra E is dibaric if and only if one of the following condition hold:

(i) $b = d = 0$ *and* $ac < 0$;

(ii) $b \neq 0, \quad ad = bc, \quad D \geqslant 0$ *and* $B^2 + C^2 \neq 0$, *where*

$D = (8a-1)^2 - 32(bd + a^2), \quad B = 4a^2 + 4bd - a + a\sqrt{D}$ *and*

$B = 4a^2 + 4bd - a - a\sqrt{D}$.

Dibaricity of CEAs

Theorem

$$E_1^{[s,t]} - \begin{cases} \text{is not dibaric evolution algebra for all } (s,t) \in \{(s,t) : s \leqslant t < a\} \\ \text{is dibaric evolution algebra for all } (s,t) \in \{(s,t) : t \geqslant a\} \end{cases};$$

$E_2^{[s,t]}$ *is not dibaric evolution algebra for any* $s, t \in \mathcal{T}$;

$$E_3^{[s,t]} - \begin{cases} \text{is not dibaric evolution algebra for all} \\ (s,t) \in \{(s,t) : s \leqslant t < a, \ h^2(s)g^2(s) < 1\}; \\ \text{is dibaric evolution algebra for all} \\ (s,t) \in \{(s,t) : s \leqslant t < a, \ h^2(s)g^2(s) > 1\}; \\ \text{is not dibaric evolution algebra for all} \\ (s,t) \in \{(s,t) : t \geqslant a, \ h(s)g(s) = \pm 1\}; \end{cases}$$

Theorem

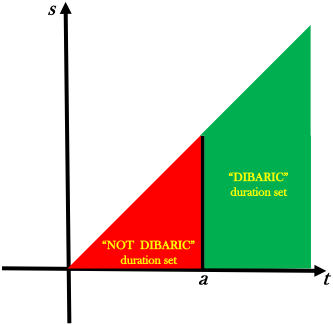
$E_4^{[s,t]}$ is not dibaric evolution algebra for any $s, t \in \mathcal{T}$;

$$E_5^{[s,t]} - \begin{cases} \begin{array}{c} \textit{is not dibaric evolution algebra for all} \\ (s,t) \in \{(s,t) : s \leqslant t < b\} \end{array} ; \\ \begin{array}{c} \textit{is not dibaric evolution algebra for all} \\ (s,t) \in \left\{(s,t) : t \geqslant b, \frac{\Phi^2(t)w(s)}{\Phi(s)} > 0\right\} \end{array} ; \\ \begin{array}{c} \textit{is dibaric evolution algebra for all} \\ (s,t) \in \left\{(s,t) : t \geqslant b, \frac{\Phi^2(t)w(s)}{\Phi(s)} < 0\right\} \end{array} ; \end{cases}$$

Graphic view of property transition

Theorem

$$E_1^{[s,t]} - \begin{cases} \textit{is not dibaric evolution algebra for all } (s,t) \in \{(s,t) : s \leqslant t < a\} \\ \textit{is dibaric evolution algebra for all } (s,t) \in \{(s,t) : t \geqslant a\} \end{cases};$$



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