Involutions gradings and identities on matrix algebras

Thiago Castilho de Mello[†]

February 21, 2018

First International Workshop
Non-associative Algebras in Cádiz

† Universidade Federal de São Paulo
Supported by CNPq and Fapesp

Gradings on matrix algebras

Gradings and graded identities

Gradings on algebras

If A is an F-algebra and G is a group, A is G-graded if

$$A = \bigoplus_{g \in G} A_g \qquad \text{and} \qquad A_g A_h \subseteq A_{gh}, \quad \text{for} \quad g, h \in G.$$

If $x \in A_g$, for some $g \in G$, we say that x is homogeneous of degree g.

Free G-graded algebra

$$X = \bigcup_{g \in G} X_g \qquad x \in X_g, \deg(x) = g$$

$$F\langle X \mid G\rangle = F\langle X\rangle$$

Graded identities

A polynomial $f(x_1, \ldots, x_n) \in F\langle X \mid G\rangle$ is a G-graded identity of a G-graded algebra A, if $f(a_1, \ldots, a_n) = 0$, for any $a_i \in A$ such that $\deg(a_i) = \deg(x_i)$.

Examples on matrix algebras

2×2 matrices

If $M_2(F)_0 = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ and $M_2(F)_1 = \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}$

$M_2(F) = M_2(F)_0 \oplus M_2(F)_1$ is a $\mathbb{Z}_2$-grading on $M_2(F)$.

Identities

(1992) Di Vincenzo: Identities follow from $[y_1, y_2]$ and $z_1 z_2 z_3 - z_3 z_2 z_1$.

Examples on matrix algebras

$n \times n$ matrices

$$\text{If } M_n(F)_t = \begin{array}{c} \begin{array}{cccccc} & & & t+1 & & \end{array} \\ \left(\begin{array}{cccccc} 0 & \cdots & 0 & F & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & F \\ F & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & F & 0 & \cdots & 0 \end{array}\right) \end{array}$$

then $M_n(F) = \bigoplus_{t \in \mathbb{Z}_n} M_n(F)_t$ is a $\mathbb{Z}_n$-grading.

Identities

(1999) Vasilovsky ($\mathrm{char}(F) = 0$)

(2002) Azevedo (arbitrary infinite field): Identities follow from:

$[x_1, x_2] = 0, \quad \deg(x_1) = \deg(x_2) = 0;$

$x_1x_2x_3 - x_3x_2x_1 = 0, \quad \deg(x_1) = \deg(x_3) = -\deg(x_2).$

Types of Gradings

Elementary gradings

- G a group
- $(g_1, \ldots, g_n) \in G^n$
- If $g \in G$, define $R_g = \mathrm{span}\{e_{ij} \mid g_i^{-1} g_j = g\}$

Then $M_n(F) = \bigoplus_{g \in G} R_g$ is a G-grading on $M_n(F)$ called *elementary grading defined by* $(g_1, \ldots, g_n)$.

Theorem
If G is any group, a G-grading of $M_n(F)$ is elementary if and only if all matrix units e_{ij} are homogeneous.

Fine gradings

A G-grading on A is a *fine grading* if $\dim A_g \leq 1$, for all $g \in G$.

Types of Gradings

Pauli gradings (or ε-gradings)
If $\varepsilon \in F$ is a primitive n-th root of 1, define the $n \times n$ matrices over F:

$$X_a = \begin{pmatrix} \varepsilon^{n-1} & 0 & \dots & 0 & 0 \\ 0 & \varepsilon^{n-2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \varepsilon & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{and} \quad X_b = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

If $g = (\bar{i}, \bar{j}) \in G = \mathbb{Z}_n \times \mathbb{Z}_n$, let $C_g = X_a^i X_b^j$ and denote $R_g = span_F\{C_g\}$.

Then $R = M_n(F) = \bigoplus_{g \in G} R_g$ is a G-grading on R.

This grading is called *ε-grading* or *Pauli grading*.

Gradings on tensor products

Induced gradings

- Let $A = \oplus_{g\in G} A_g$ be any G-graded algebra.
- Let $B = M_n(F) = \bigoplus_{g\in G} B_g$ with an elementary grading induced by $(g_1, \ldots, g_n)$.
- Set
$$(A \otimes B)_g = \mathrm{span}_F\{a \otimes e_{ij} | a \in A_h, g_i^{-1} h g_j = g\}.$$

Then $A \otimes B$ is G-graded and such grading is called *induced*.

Gradings on Matrix Algebras

Theorem (Bahturin, Segal and Zaicev)
Let $R = M_n(F) = \bigoplus_{g \in G} R_g$ be a G-graded matrix algebra over F. Then there exists a decomposition $n = kl$, a subgroup $H \subseteq G$ of order k^2 and a l-tuple $\overline{g} = (g_1, \ldots, g_l) \in G^l$ such that $M_n(F)$ is isomorphic as a G-graded algebra to the tensor product $M_k(F) \otimes M_l(F)$ with an induced G-grading where $M_k(F)$ is a H-graded algebra with fine H-grading and $M_l(F)$ is endowed with an elementary grading determined by $\overline{g}$. Moreover, H decomposes as $H \cong H_1 \times \cdots \times H_t$, $H_i \cong \mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i}$ and $M_k(F)$ is isomorphic to $M_{n_1}(F) \otimes \cdots \otimes M_{n_t}(F)$ as an H-graded algebra, where $M_{n_i}(F)$ is an H_i-graded algebra with some ε_i-grading.

Identities with elementary gradings

(2002) Bahturin and Drensky: $M_n(F)$ with elementary grading defined by $(g_1, \dots, g_n)$, with $g_i \neq g_j$, $i \neq j$. $\mathrm{Char}(F) = 0$.
Identities follow from:
$[x_1, x_2] = 0, \quad \deg(x_1) = \deg(x_2) = 0;$
$x_1x_2x_3 - x_3x_2x_1 = 0, \quad \deg(x_1) = \deg(x_3) = -\deg(x_2).$
Monomial identities of degree up to $4s^{2s+2}$, $s = |G_0|$.

(2013) Diniz: Infinite field of $\mathrm{char}(F) = p > 0$

(2015, 2016) Centrone, de M., Diniz: Monomial identities follows from identities of degree up to $2n - 1$. Conjucture: monomial identities of degree up to n.

(2017) Centrone, de M., Diniz: Identities above and monomial identities of degree up to n (the minimum possible).

Involutions on matrix algebras

Involutions on matrix algebras

An involution on an F-algebra A is an antiautomorphism of order two, i.e., a linear isomorphism $^* : A \longrightarrow A$ satisfying for all a, $b \in A$,

$$(ab)^* = b^* a^* \text{ and } (a^*)^* = a.$$

Examples

$$\begin{array}{ccc} M_n(F) & \longrightarrow & M_n(F) \\ A & \longmapsto & A^t \end{array}$$

$$\begin{array}{ccc} M_{2n}(F) & \longrightarrow & M_{2n}(F) \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \longmapsto & \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} \end{array}$$

Description of involutions on matrix algebras:

Theorem
Let $*$ be an involution on $M_n(F)$. Then for every $X \in M_n(F)$,

$$X^* = \Phi^{-1} X^t \Phi,$$

for some non-singular matrix Φ which is either symmetric or skew-symmetric. Moreover, Φ is uniquely defined by $*$ up to a scalar factor.

Identities with involutions on matrix algebras

(2005) Giambruno and Zaicev:

Theorem
Let F be an infinite field, $char(F) \neq 2$ and let $*$ be an involution on $M_n(F)$. Then

$$Id(M_n(F), *) = Id(M_n(F), t) \text{ or } Id(M_n(F), *) = Id(M_n(F), s).$$

The second possibility can occur only if n is even.

Involutions and gradings on matrix algebras

Graded identities with involution

(2016) Haile and Natapov: Identities of $M_n(\mathbb{C})$ with transpose involution and elemetary G-grading induced by $(g_1, \ldots, g_n)$, where $G = \{g_1, \ldots, g_n\}$ follow from:

$x_{i,e} - x_{i,e}^*$

$x_{i,e}x_{j,e} - x_{j,e}x_{i,e}$

(Graph theory technique)

Remark

The identity $x_1x_2x_3 - x_3x_2x_1$, $\deg(x_1) = \deg(x_3) = \deg(x_2)^{-1}$ is a consequence of the identities above.

Free object?

$F\langle X \mid (G, *)\rangle = F\langle X \cup X^*\rangle$.

$X = \bigcup_{g\in G} X_g$

$\deg(x_g^*) = \deg(x_g)^{-1}$

Degree-inverting involutions on matrix algebras

Example
If $M_n(F)$ is endowed with any elementary grading defined by $(g_1, \ldots, g_n)$,

$$\deg(e_{ij}^t) = \deg(e_{ji}) = g_j^{-1} g_i = (g_i^{-1} g_j)^{-1} = \deg(e_{ij})^{-1}$$

Degree-inverting involution
An involution $*$ on a G-graded algebra A is a degree-inverting involution if for all $g \in G$, $A_g^* \subseteq A_{g^{-1}}$.

Problem
Describe the degree-inverting involutions on $M_n(F)$.

Remark
Involutions satisfying $A_g^* \subseteq A_g$ are called graded involutions and have been described by Bahturin Shestakov and Zaicev in 2005.
Such description was applied for classifying gradings on simple finite dimensional Lie and Jordan algebras.

Elementary grading with involution

Lemma
Let $R = M_n(F) = \oplus_{g\in G} R_g$ with an elementary grading be a matrix algebra with a degree-inverting involution $*$, then R is isomorphic to $M_n(F)$ with an elementary G-grading defined by an n-tuple $(g_1, \ldots, g_n)$ and with involution $X^* = \Phi^{-1}X^t\Phi$, where

1. $n = 2l + m$, for some l, $m \in \mathbb{N}$, and $\Phi = \begin{pmatrix} 0 & I_l & 0 \\ I_l & 0 & 0 \\ 0 & 0 & I_m \end{pmatrix}$,

   if $*$ corresponds to a symmetric matrix. Moreover,

   1.1 if $m = 0$, then after a renumbering, $g_1 g_{l+1}^{-1} = \cdots = g_l g_{2l}^{-1}$, and $g_i^2 = g_{i+l}^2$, for all $i \in \{1, \ldots, l\}$;

   1.2 if $l \neq 0$ and $m \neq 0$, then after a renumbering $g_1 g_{l+1}^{-1} = \cdots = g_k g_{2l}^{-1}$, $g_1^2 = g_2^2 = \cdots = g_{2l}^2$, and $g_1 g_{l+1} = g_2 g_{l+2} = \cdots = g_l g_{2l} = g_{2l+1}^2 = \cdots = g_m^2$;

   1.3 if $l = 0$, $*$ is the transpose involution and $g_1, \ldots, g_m$ are arbitrary.

Elementary grading with involution

Lemma
Let $R = M_n(F) = \oplus_{g \in G} R_g$ with an elementary grading be a matrix algebra with a degree-inverting involution $*$, then R is isomorphic to $M_n(F)$ with an elementary G-grading defined by an n-tuple $(g_1, \ldots, g_n)$ and with involution $X^* = \Phi^{-1} X^t \Phi$, where

2. $n = 2l$, for some $l \in \mathbb{N}$, and $\Phi = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$.

   if $*$ corresponds to a skew-symmetric matrix. Moreover, after a renumbering $g_1 g_{l+1}^{-1} = \cdots = g_l g_{2l}^{-1}$, and $g_i^2 = g_{i+l}^2$, for all $i \in \{1, \ldots, l\}$.

Lemma
Let $R = M_n(F)$, $n \geq 2$ with an ε-grading $R = \bigoplus_{g\in G} R_g$. If $*$ is a degree-inverting involution, then $n = 2$ and $*$ is given by $X^* = \Phi^{-1}X^t\Phi$, where Φ is a scalar multiple of one of the matrices I, X_a, X_b or X_aX_b.

Degree-inverting involutions on matrix algebras

Theorem
Let $R = M_n(F) = \oplus_{g\in G} R_g$ with a degree-inverting involution $*$. Then R is isomorphic as a G-graded algebra to the tensor product $R^{(0)} \otimes R^{(1)} \otimes \cdots \otimes R^{(k)}$ of a matrix subalgebra $R^{(0)}$ with elementary grading and $R^{(1)} \otimes \cdots \otimes R^{(k)}$ a matrix subalgebra with fine grading. Suppose further that both these subalgebras are invariant under the involution $*$. Then $n = 2^k m$ and

1. *$R^{(0)} = M_m(F)$, with an elementary G-grading defined by an m-tuple $\overline{g} = (g_1, \ldots, g_m)$ of elements of G. The involution $*$ acts on $M_m(F)$ as $X^* = \Phi^{-1} X^t \Phi$, where Φ and the elements $g_1, \ldots, g_m$ are as in the previous lemmas.*
2. *$R^{(1)} \otimes \cdots \otimes R^{(k)}$ is a $T = T_1 \times \cdots \times T_k$-graded algebra and any $R^{(i)} \cong M_2(F)$ is $T_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra. The involution $*$ acts on $R^{(1)} \otimes \cdots \otimes R^{(k)}$ as in the previous Lemma.*

Identities with the transpose involution

Theorem
Let $M_n(F)$ be a matrix algebra over an infinite field, endowed with the transpose involution and with an elementary G-grading induced by $(g_1, \dots, g_n)$ with $g_i \neq g_j$ for $i \neq j$. The graded identities of $M_n(F)$ with the transpose involution follow from

$x_{i,e} - x_{i,e}^*$

$x_{i,e}x_{j,e} - x_{j,e}x_{i,e}$

Graded monomial identities of degree up to $2n - 1$.

Gradings and graded identities on tensor products

Gradings on tensor products

If A is G-graded and B is H-graded, one can define a $G \times H$-grading on $A \otimes B$ by:

$$(A \otimes B)_{(g,h)} = A_g \otimes B_h$$

Example

If $M_n(F)$ is G-graded, then $M_n(E) \cong M_n(F) \otimes E$ is $G \times \mathbb{Z}_2$-graded.

Problem

Find a basis for the $G \times \mathbb{Z}_2$-graded identities for $A \otimes E$ from a basis of G-graded identities of A.

Tensor products by the Grassmann algebra

Let us consider the sets

- $Z = X' \bigcup Y'$, where
- $X' = \bigcup_{g \in G} X'_g$ (even variables) and $Y' = \bigcup_{g \in G} Y'_g$ (odd variables)

We work on the free $G \times \mathbb{Z}_2$- algebra $F\langle Z \rangle$.

For each $J \subseteq \mathbb{N}$, we define

$$\begin{array}{rccc} \varphi_J : & F\langle X \rangle & \longrightarrow & F\langle Z \rangle \\ & x_g^{(i)} & \longmapsto & \begin{cases} x_g^{(i)}, & \text{if } i \notin J \\ y_g^{(i)}, & \text{if } i \in J \end{cases} \end{array}$$

If m is a multilinear monomial in $F\langle Z \rangle$, we can write:

$m = m_0 y_{\sigma(i_1)} m_1 y_{\sigma(i_2)} \cdots y_{\sigma(i_k)} m_k$, where $i_1 < i_2 < \cdots < i_k$, and define

$$\zeta(m) = (-1)^{\sigma} m.$$

Tensor products by the Grassmann algebra

Now for each $J \subseteq \mathbb{N}$, and a multilinear polynomial f in $F\langle X \mid G \rangle$, we define

$$\zeta_J(f) = \zeta(\varphi_J(f)).$$

Theorem (Di Vincenzo and Nardozza)
Let A be a G-graded algebra and $\mathcal{E} \subset F\langle X|G\rangle$ be a system of multilinear generators for $Id_G(A)$. Then the set

$$\{\zeta_J(f) \mid f \in \mathcal{E},\ J \subseteq \mathbb{N}\}$$

is system of multilinear generators of $Id_{G\times\mathbb{Z}_2}(A \otimes E)$

Color commutative superalgebras

The map $\beta : H \times H \longrightarrow F^{\times}$ is a skew-symmetric bicharacter of H if

- $\beta(g+h,k) = \beta(g,k)\beta(h,k)$.
- $\beta(g,h) = \beta(h,g)^{-1}$.

Let $C = \bigoplus_{h\in H} C_h$ be a graded algebra. If $x \in C_g$ and $y \in C_h$, we define

$$[x,y]_\beta := xy - \beta(g,h)yx.$$

and extend it to C by linearity.

If C satisfies $[x,y]_\beta \equiv 0$, we say that C is a **color commutative superalgebra**.

Color commutative superalgebras

Example

1. Let $\beta : \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow F^\times$ be defined as $\beta(g,h) = 1$, if $g = 0$ or $h = 0$ and $\beta(1,1) = -1$.
   Then one can simply verify that E satisfies $[x,y]_\beta \equiv 0$.
2. Let $H = \mathbb{Z}_n \times \mathbb{Z}_n$ and let $\beta : H \times H \longrightarrow F^\times$ given by

$$\beta((k,l),(r,s)) = \varepsilon^{rl-ks}.$$

   Then $M_n(F)$ with the Pauli grading satisfies $[x,y]_\beta \equiv 0$, since

$$X_a X_b = \varepsilon X_b X_a.$$

E and $M_n(F)$ are color commutative superalgebras.

Problem

If C is an H-graded color commutative superalgebra, find a basis for the $G \times H$-graded identities of $A \otimes C$ from a basis of G-graded identities of A.

Let us now consider the free algebra $F\langle X \mid G \times H\rangle$. Let us denote its variables by $x_{(g,h)}^{(i)}$, for $g \in G, h \in H$ and $i \in \mathbb{N}$.

For each sequence of elements of H, $\mathbf{h} = (h_i)_{i\in\mathbb{N}}$, we define a map $\varphi_{\mathbf{h}} : F\langle X|G\rangle \longrightarrow F\langle X|G \times H\rangle$ as the unique homomorphism of G-graded algebras satisfying $\varphi_{\mathbf{h}}(x_{g_i}^{(i)}) = x_{(g_i,h_i)}^{(i)}$.

Tensor products by color commutative superalgebras

Let $m = x^{(i_{\sigma(1)})}_{(g_{i_{\sigma(1)}} h_{i_{\sigma(1)}})} \cdots x^{(i_{\sigma(k)})}_{(g_{i_{\sigma(k)}} h_{i_{\sigma(k)}})}$ be multilinear with $i_1 < \cdots < i_k$.

If in the free H-graded color commutative superalgebra, we have

$$x^{(i_1)}_{h_{i_1}} \cdots x^{(i_k)}_{h_{i_k}} = \lambda_{\mathbf{h}\sigma} x^{(i_{\sigma(1)})}_{h_{i_{\sigma(1)}}} \cdots x^{(i_{\sigma(k)})}_{h_{i_{\sigma(k)}}},$$

with $\lambda_{\mathbf{h}\sigma} \in F^{\times}$, we define $\zeta(m) = \lambda_{\mathbf{h}\sigma} m$.

Now for each sequence $\mathbf{h} = (h_i)_{i \in \mathbb{N}}$, and a multilinear polynomial f in $F\langle X \mid G \rangle$, we define

$$\zeta_{\mathbf{h}}(f) = \zeta(\varphi_{\mathbf{h}}(f)).$$

Theorem (Bahturin-Drensky)
The polynomial $f(x^{(i_1)}_{g_1}, \ldots, x^{(i_m)}_{g_m})$ is a multilinear G-graded identity of A if and only if $\zeta(\varphi_{\mathbf{h}}(f))$ is a multilinear $G \times H$-graded identity of $A \otimes C$.

Tensor products by color commutative superalgebras

Theorem (Diniz, de Mello)
Let C be an H-graded color commutative superalgebra, generating the variety of all H-graded color commutative superalgebras and let R be any G-graded algebra. If $\mathcal{E}$ is a system of multilinear generators for $Id_G(R)$, then the set

$$S = \{\zeta_{\mathbf{h}}(f) \,|\, f \in \mathcal{E}, \mathbf{h} \in H^n, n \in \mathbb{N}\}$$

is a system of multilinear generators of $Id_{G\times H}(R \otimes C)$.

Applications

- $G \times \mathbb{Z}_2$-graded identities of $UT(d_1, \ldots, d_m; E)$
- $G \times H$-identities of $UT_n(F) \otimes M_t(F)$.
- Generators for central polynomials (Diniz, Fidelis and Mota, 2017).

Thanks