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A categorical characterisation of Lie algebras

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(LACC)

Definition (Gray, 2012)

Let $\mathcal{C}$ be a category with finite limits and zero object.
$\mathcal{C}$ is locally algebraically cartesian closed ((LACC) for short) if and only if the kernel functor

$$\mathfrak{i}_B^*\colon\ \mathtt{Pt}_B(\mathcal{C}) \longrightarrow \mathcal{C}$$

$$\begin{array}{c} X \\ s \uparrow \downarrow f \\ B \end{array}$$

has a right adjoint for all B.

Examples

Groups, Lie algebras and abelian categories.

Non-associative algebras

Definition

Let $\mathbb{K}$ be a field. A non-associative algebra is a $\mathbb{K}$-vector space with a linear map

$$A \otimes A \to A.$$

We denote the category by $\mathtt{Alg}_{\mathbb{K}}$.

A *subvariety* of $\mathtt{Alg}_{\mathbb{K}}$ is any equationally defined class of algebras, considered as a full subcategory $\mathcal{V}$ of $\mathtt{Alg}_{\mathbb{K}}$.

Non-associative algebras

Theorem (Zhevlakov, Slin'ko, Shestakov, Shirshov)

If $\mathcal{V}$ is a variety of algebras over an infinte field $\mathbb{K}$, all of its laws are of the form $\phi(x_1, \ldots, x_n)$, where ϕ is a non-associative polynomial.
Moreover, each of its homogeneous components $\psi(x_{i_1}, \ldots, x_{i_n})$ is also a law.

This means that if

$$(xy)z + x^2$$

is a law of $\mathcal{V}$, then

$$x(yz) + y(zx) + z(xy) + xy + yx$$

is a law of $\mathcal{V}$, then

$$x(yz)$$
$$x^2$$

Constructions in varieties algebras (informally)

Let $\mathcal{V}$ be a variety of algebras, the free $\mathcal{V}$-algebra generated by a set $X = \{x, y, z, \dots\}$ is the $\mathbb{K}$-vector space formed by

$$\begin{array}{l} x, y, z, \dots \\ (xx), (xy), (xz), (yz), \dots \\ (x(xx)), ((xx)x), (x(yz)), (y(xz)), \dots \\ \vdots \end{array}$$

quotiened by the equations of $\mathcal{V}$ and multiplication is juxtaposition.

The coproduct in $\mathcal{V}$ of two algebras A and B, is the free algebra generated by A and B where if we find a word formed by just elements from on of them, we substitute by its multiplication.

Points and actions

Let $X, B \in \mathcal{V}$. Consider the split extension

$$B\flat X \longrightarrow B + X \underset{(1\ 0)}{\overset{\iota_1}{\leftrightarrows}} B$$

Definition (Bourn-Janelidze, 1998)

An action of B on X (or a B-action) is a morphism $\xi\colon B\flat X \to X$, satisfying some monadicity properties.

Theorem (Bourn-Janelidze, 1998)

There is an equivalence of categories

$$\mathtt{Pt}_B(\mathcal{V}) \simeq B\text{-}\mathtt{Act}(\mathcal{V})$$

Forgetful functor

Proposition

Let $\mathcal{C}$ be a category with finite limits and zero object.
$\mathcal{C}$ is (LACC) *if and only if the kernel functor*

$$\mathfrak{i}_B^* \colon B\text{-}\mathtt{Act} \simeq \mathtt{Pt}_B(\mathcal{V}) \to \mathcal{C}$$

$$\begin{array}{c} B\flat X \\ \downarrow \\ X \end{array} \longmapsto X$$

has a right adjoint for all B.

Preservation of $B\flat(-)$

Proposition

Let $\mathcal{V}$ be a variety of non-associative algebras.
It is (LACC) *if and only if the cannonical comparison*

$$(B\flat X + B\flat Y) \to B\flat(X + Y)$$

is an isomorphism.

Algebraic coherence

Theorem

The following are equivalent:

- $\mathcal{V}$ *is algebraic coherent,*
  i. e. the map $(B\flat X + B\flat Y) \to B\flat(X+Y)$ *is surjective.*
- *There exist* $\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8 \in \mathbb{K}$ *such that*

$$z(xy) = \lambda_1(zx)y + \lambda_2(zy)x \cdots \lambda_8 y(xz)$$
$$(xy)z = \mu_1(zx)y + \mu_2(zy)x \cdots \mu_8 y(xz)$$

  are laws of $\mathcal{V}$*.*
- *For any ideal* I*,* I^2 *is also an ideal (i.e.* $\mathcal{V}$ *is a 2-variety).*
- $\mathcal{V}$ *is an Orzech category of interest.*

Nilpotent algebras

Proposition

If $\mathcal{V}$ is (LACC) and $x(yz) = 0$ is a law in $\mathcal{V}$, then $\mathcal{V}$ is abelian.

Proof: Let B, X, Y be free algebras on one generator. Since $\mathcal{V}$ is (LACC), the morphism

$$(B\flat X + B\flat Y) \to B\flat(X+Y)$$

is an isomorphism.

The element

$$x(yb) \in B\flat(X+Y)$$

comes from zero.

The expression yb plays the role of just "one element" in $B\flat(X+Y)$.

Then if $x(yb)$ is zero, or $x(yb) = 0$ or $yb = 0$ have to be rules of $\mathcal{V}$. In both cases, it implies that the algebra is abelian.

Associative algebras

Proposition

The variety of associative algebras is not (LACC).

Proof: Consider again B, X, Y as free algebras on one generator. Assume that we have an isomorphism:

$$(B\flat X + B\flat Y) \to B\flat(X + Y)$$

Then $(xb)y$ and $x(by)$ go to the same element in $B\flat(X + Y)$ but they are different in $(B\flat X + B\flat Y)$.

Leibniz algebras

Proposition

The variety of Leibniz algebras is not (LACC).

Proof: In the Leibniz case, we have they identities

$$b(xy) = (bx)y + x(by)$$

$$b(xy) = -(xb)y + x(by)$$

Then, we have that $(bx)y + (xb)y = 0$.
Again, $(bx)y + (xb)y$ is zero in $B\flat(X + Y)$ but it does not need to be in $(B\flat X + B\flat Y)$.

Operations of degree 2

Theorem

If $\mathcal{V}$ is a $\mathrm{(LACC)}$ commutative variety of algebras, i. e. $xy = yx$ is a law, then $\mathcal{V}$ is abelian.

Theorem

If $\mathcal{V}$ is a $\mathrm{(LACC)}$ anticommutative variety of algebras, i. e. $xy = -yx$ is a law, then $\mathcal{V}$ is subvariety of $\mathtt{Lie}_{\mathbb{K}}$.

Non-commutative and non-anticommutative

Let us assume that there are no operations of degree 2.
We need to see if there is any variety such that the map

$$(B\flat X + B\flat Y) \to B\flat(X + Y)$$

is an isomorphism.

$$\begin{aligned} x(by) &= \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx) \\ &+ \lambda_5(xy)b + \lambda_6(yx)b + \lambda_7 b(xy) + \lambda_8 b(yx) \end{aligned}$$

$$\begin{aligned} &= \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx) \\ &+ \lambda_5\big(\mu_1(bx)y + \mu_2(xb)y + \mu_3 y(bx) + \cdots + \mu_7 x(by) + \mu_8 x(yb)\big) \\ &+ \lambda_6\big(\mu_1(by)x + \mu_2(yb)x + \mu_3 x(by) + \cdots + \mu_7 y(bx) + \mu_8 y(xb)\big) \\ &+ \lambda_7\big(\lambda_1(bx)y + \lambda_2(xb)y + \lambda_3 y(bx) + \cdots + \lambda_7 x(by) + \lambda_8 x(yb)\big) \\ &+ \lambda_8\big(\lambda_1(by)x + \lambda_2(yb)x + \lambda_3 x(by) + \cdots + \lambda_7 y(bx) + \mu_8 y(xb)\big) \end{aligned}$$

Non-commutative and non-anticommutative

After obtaining 128 different polynomials that the coefficients $\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8$ have to satisfy, we found using Gröbner basis that there is no possible combination. Therefore,

Theorem

If $\mathcal{V}$ is a (LACC) variety of non-associative algebras without any law of order 2, then $\mathcal{V}$ is abelian.

Subvarieties of Lie and *n*-arity

Theorem

If $\mathcal{V}$ *is a proper* (LACC) *subvariety of* $\mathtt{Lie}_{\mathbb{K}}$, *then it is abelian.*

Theorem

If $\mathcal{V}$ *is a* (LACC) *variety of n-algebras, with* $n \neq 2$, *then it is abelian.*

Main theorem

Theorem

Let $\mathcal{V}$ *be a non-abelian* (LACC) *variety of non-associative n-algebras.*
Then, $\mathcal{V} = \mathtt{Lie}_{\mathbb{K}}$.

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